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Third-order resonance effects and the nonlinear stability of drop oscillations

By RAMESH NATARAJAN AND ROBERT A. BROWN
Department of Chemical Engineering and Materials Processing Center,
Massachusetts Institute of Technology,
Cambridge, MA 02139, USA

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The three-dimensional nonlinear oscillations of an isolated, inviscid drop with surface tension are studied by a multiple timescale analysis and pre-averaging applied to the variational principle for the appropriate Lagrangian. Amplitude equations are derived which describe the generic cubic resonance caused by the spatial degeneracy of the eigenfrequencies of the linear normal modes. This resonant coupling leads to the instability of the finite amplitude axisymmetric oscillations to small non-axisymmetric perturbations, as is demonstrated here for the three- and four-lobed normal modes. Solutions to the interaction equations that describe finite amplitude, non-axisymmetric travelling-wave solutions are also obtained and their stability is investigated. A non-generic cubic resonance between the two-lobed and four-lobed oscillatory modes leads to quasi-periodic motions.

1. Introduction

Rayleigh (1879) derived the frequencies for the small-amplitude oscillations of an isolated inviscid drop about the spherical equilibrium shape. The deformations in the drop shape were described in terms of the linear normal modes $S_n(\theta,\varphi)\,e^{i\omega_n t}$, where $S_n(\theta,\varphi)$ is a surface spherical harmonic of degree n and ω_n, the corresponding frequency, is given by

$$\omega_n^2 = \frac{\sigma}{\rho R^3}n(n-1)(n+2), \tag{1.1}$$

where σ is the surface tension of the drop, ρ is the density and R is the drop radius. The oscillation modes $n=0$ and $n=1$ are excluded by the requirements of conservation of mass and linear momentum in a spherical coordinate system with the origin fixed at the centre of mass of the drop. Rayleigh's results for this problem and on the cognate problems of an oscillating bubble and an oscillating drop immersed in an inviscid suspending medium are discussed in Lamb (1932).

Subsequent linear analyses have included the viscosity of the drop (Reid 1960), the viscous effects of the suspending medium (Miller & Scriven 1968), as well as the dynamical evolution of the drop from general initial perturbations (Prosperetti 1981). Experimental measurements of the oscillation frequencies by Marston & Apfel (1979), Trinh, Zwern & Wang (1982) and Trinh & Wang (1982) agree reasonably with the calculations of these linear theories but also show some new phenomena that can only be accounted for by the analysis of large-amplitude oscillations. For example, Trinh & Wang (1982) show that the oscillation frequency of drops suspended in a neutrally buoyant liquid decreases with increasing amplitude.

Tsamopoulos & Brown (1983, 1984) have calculated this decrease for the moderate amplitude, axisymmetric oscillations of an isolated, inviscid drop from a nonlinear analysis.

The present work is motivated in part by some other experimental observations of Trinh & Wang (1982). In their work, axisymmetric drop oscillations were excited and maintained by a suitably imposed acoustic pressure field. They observed that it was difficult to maintain a large-amplitude oscillation because of a tendency for a non-axisymmetric running wave to develop on the drop surface; this phenomenon eventually lead to the drop rotating as a solid body. These transitions have been documented by Trinh & Wang in a striking set of photographs. Similar difficulties with establishing large-amplitude axisymmetric oscillations due to the onset of rotational motion were encountered by Jacobi *et al.* (1982) in experiments on a freely levitated drop in a low-gravity environment aboard a rocket flight. As shown in this report, such observations can be explained as being due to a third-order resonant instability of the axisymmetric motions to small non-axisymmetric perturbations that exist due to imperfections in the experiments.

The motion of an inviscid drop is a complicated free-boundary problem with nonlinearities arising from inertia, capillarity and the coupling of the surface kinematics to the velocity field within the drop. A nonlinear analysis must account for these effects systematically as the amplitude of the oscillation is increased. The parameter ϵ, is a measure of the amplitude of the oscillation relative to the equilibrium radius of the drop and characterizes the degree of nonlinearity. Expansions are written for the dependent variables, e.g. the velocity potential, pressure and drop shape as a perturbation series in powers of ϵ. Substituting these expansions into the governing equations and separating terms of equal order in ϵ lead to a sequence of linear, inhomogenous problems with the inhomogeneities at a given order being determined from the solutions of the lower-order problems. The leading-order equations are homogenous and for this set the solutions are constructed by the usual methods of normal mode analysis and superposition. Nonlinear effects are calculated from the higher-order problems by successive substitution. As is well known, the appearance of secular terms, i.e. terms that have the same spatial form and frequency as one of the linear modes, in the higher-order problems leads to difficulties in this approach. Such terms give rise to solutions with a polynomial growth in time that render the ordering assumptions of the perturbation theory invalid after short times. Secular perturbation methods (Nayfeh & Mook 1979) show that in a single-degree-of-freedom nonlinear oscillator the appearance of such terms is due to the dependence of the nonlinear frequency on the oscillation amplitude. These methods have been used by Tsamopoulos & Brown (1983, 1984) to calculate amplitude-dependent corrections to the linear frequency for the various axisymmetric modes of drop oscillations. In these problems the resonant terms appear at third order in the amplitude and are caused by the cubic self-interaction of the primary oscillation as well as by the interaction of the primary oscillation with the bound harmonics generated by second-order corrections.

Resonance involving two or more linear normal modes is possible whenever the frequencies of these modes are commensurate, i.e. when a linear integer combination of the normal mode frequencies is zero (there also will be conditions that must be satisfied by the spatial forms of the normal modes but these are assumed to hold for the purposes of discussion). The presence of such resonances is of interest since it implies a coupling that permits the transfer of energy and angular momentum between the normal modes in addition to usual amplitude-dependent frequency shifts discussed earlier. The dynamical effects of these internal resonances may be investigated by allowing for the amplitudes and phases of the primary modes to be functions of a slow timescale that is chosen to be consistent with the order of the nonlinearity at which the resonant terms appear.

In the present context, we note that there is a $(2n+1)$ spatial degeneracy associated with the eigenfrequency given in (1.1). For example, the normal mode with $n = 3$ has seven spatial components (1 zonal harmonic, 2 tesseral harmonics of rank one, 2 tesseral harmonics of rank two and 2 sectorial harmonics) all of which have the same linear frequency. This degeneracy gives rise to an internal resonance that appears through the third-order terms in the nonlinearity and leads to a weakly nonlinear mutual interaction between these seven independent spatial components. The use of secular perturbation methods in this work to remove the resonant terms that would appear in a successive approximation procedure shows that the timescale for this mutual interaction is $O(\epsilon^2)$. The details of the internal resonance depend on the nature of the nonlinearity arising for each normal mode; nevertheless the phenomenon is generic and hence crucial to the understanding of the nature of the long-term dynamics of drop oscillations.

From another point of view, the drop shape associated with the oscillation at the frequency ω_n is not determined by the linear analysis. This is because an arbitrary linear combination of the $(2n+1)$ components of the surface harmonic $S_n(\theta, \varphi)$ may be superposed. However, not all these linear combinations are admissible finite-amplitude periodic solutions. It is only for certain special sets of initial conditions that the interaction equations will yield time-periodic solutions that resemble the linear theory but with corrections to the linear frequency that are amplitude dependent. Furthermore, to be physically realized these finite-amplitude time-periodic solutions must be stable to all possible perturbations. The internal resonance enables an energy exchange and hence identifies a class of possible unstable perturbations as well as the timescales on which these are realized. The existence of such instabilities can be tested by examining the signs of the eigenvalues of the appropriately linearized form of the interaction equations.

More generally, the interaction equations can be numerically integrated from a given initial condition to examine the long-term dynamics. Natarajan & Brown (1986) studied the interaction equations for a quadratic resonance phenomenon that occurs in the oscillating-drop problem by numerical integrations. By simultaneously calculating the Liapunov exponents of the solution trajectories (Lichtenberg & Lieberman 1983) they showed that the normal modes participating in the resonance displayed a stochastic long-term dynamic behaviour. This phenomenon was primarily due to the large dimensionality of the set of interaction equations as a consequence of including the normal modes for describing non-axisymmetric oscillations. Reducing the dimension of the interaction equations to retain only the axisymmetric components yielded integrable non-stochastic trajectories which, however, were unstable to non-axisymmetric perturbations. The relevance of those calculations to the analysis presented here is discussed further in §5.

The primary goal of this paper is to study the cubic internal resonances that appear in inviscid drop dynamics due to the spatial degeneracy of the eigenfrequencies in (1.1). A complication arises here due to the fact that the $n = 2$ and $n = 4$ modes have frequencies that are related as

$$\omega_4 \pm 3\omega_2 = 0. \tag{1.2}$$

Hence these two normal modes are resonantly coupled by cubic nonlinearities. This coupling is a peculiarity of the present problem and introduces analytical difficulties in the calculation that obscure the generic internal resonance occurring between the components of a single normal mode. For this reason we have chosen to first present the details of the calculation for the $n = 3$ primary oscillation in §3. The case of the resonant interaction between the $n = 2$ and $n = 4$ normal modes is treated in §4.

2. Variational principle and asymptotic methods

Nonlinear analysis of inviscid drop dynamics is complicated by the tedious algebra involved in the derivation of the amplitude equations that describe the resonant interactions. Fortunately, the relevant field equations and boundary conditions for the problem admit of an underlying Lagrangian from which they may be derived using Hamilton's principle. The use of a multiple timescales approach in conjunction with the pre-averaging of the Lagrangian in the resonant environment drastically reduces the algebra and leads to a concise formulation of the interaction equations. The use of this general approach was pioneered by Whitham (1967) in the study of gravity waves and by Simmons (1969) in the study of internal resonance effects in capillary–gravity waves. The previous study of quadratic resonance effects in inviscid drop dynamics (Natarajan & Brown 1986) has already demonstrated the usefulness of the variational approach. Its utility in the present context is even greater since the internal resonance appears at a higher order in the nonlinearity and the algebra is correspondingly more involved.

In a spherical coordinate system with origin at the centre of mass of the drop, we denote $\phi(r,\theta,\varphi,t)$ as the velocity potential, $f(\theta,\varphi,t)$ as the shape of the drop surface and V as the volume of the drop. The Lagrangian for the motion is given by

$$\begin{aligned}\mathscr{L} &= \mathscr{L}(f, f_\theta, f_\varphi, \phi_r, \phi_\theta, \phi_\varphi, \phi_t, \bar{p}_0)\\ &= \rho\int_0^{2\pi}\int_0^{\pi}\int_0^{f}\left[\frac{1}{2}\left\{\phi_r^2+\frac{1}{r^2}(\phi_\theta^2+\phi_\varphi^2\csc^2\theta)\right\}-\phi_t\right]r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\\ &\quad+\sigma\int_0^{2\pi}\int_0^{\pi} f^2\left[1+\frac{f_\theta^2}{f^2}+\frac{f_\varphi^2}{f^2}\csc^2\theta\right]^{\frac{1}{2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi+\bar{p}_0\int_0^{2\pi}\int_0^{\pi}\left[\tfrac{1}{3}f^3-\frac{V}{4\pi}\right]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.\end{aligned} \quad (2.1)$$

The quantity $\bar{p}_0$ is the reference pressure difference, taken as the mean ambient pressure relative to the pressure in the equilibrium spherical drop and the functions ϕ and f are single-valued functions of the coordinates. With no further restrictions the field equations and boundary conditions appropriate to the problem follow from Hamilton's principle, which requires that

$$\delta\int_{t_0}^{t_1}\mathscr{L}\,\mathrm{d}t = 0, \quad (2.2)$$

where the limits of integration, t_0 and t_1 are arbitrary.

As noted earlier, the parameter ϵ which is a measure of the amplitude of the drop oscillation relative to the drop radius serves as an ordering parameter in the analysis. There is no physical magnitude associated with ϵ and its numerical value can be set to unity at any stage in the calculation. We take

$$f(\theta,\varphi,t) = R[1+\epsilon f'(\theta,\varphi,\xi)], \quad (2.3a)$$

$$\phi(r,\theta,\varphi,t) = \left(\frac{\sigma R}{\rho}\right)^{\frac{1}{2}}[\epsilon\phi'(\eta,\theta,\varphi,\xi)], \tag{2.3b}$$

$$\overline{p}_0(t) = \frac{\sigma}{R}[-2+\epsilon^2 p_0'(\xi)], \tag{2.3c}$$

where η, the dimensionless radial coordinate, and ξ, the dimensionless time coordinate, are normalized with the scales R and $(R^3\rho/\sigma)^{\frac{1}{2}}$ respectively. The primed quantities in (2.3) are dimensionless and represent deviations from the spherical equilibrium state. The term $p_0'(\xi)$ in (2.3c) is required to account for the fact that the reference pressure is based on the mean drop radius which has a second-order time-dependent correction arising from the nonlinearity.

Substituting (2.3) into (2.1) and expanding in a power series in ϵ we obtain for the dimensionless Lagrangian $\mathscr{L}'$ correct to terms of $O(\epsilon^5)$ as

$$\begin{aligned}
\mathscr{L}' = {}& \epsilon\left\{\int_0^{2\pi}\int_0^{\pi}\int_0^{1}(-\phi_\xi'\eta^2)\sin\theta\,\mathrm{d}\eta\,\mathrm{d}\theta\,\mathrm{d}\varphi\right\} \\
&+\epsilon^2\left\{\int_0^{2\pi}\int_0^{\pi}\left[-f'^2+\tfrac{1}{2}(f_\theta'^2+f_\varphi'^2\csc^2\theta)+\tfrac{1}{2}\int_0^1(\phi_\eta'^2\eta^2+\phi_\theta'^2+\phi_\varphi'^2\csc^2\theta)\,\mathrm{d}\eta\right.\right. \\
&\left.\left.+(-\phi_\xi'\eta^2)|_{\eta=1}f'\right]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\right\} \\
&+\epsilon^3\left\{\int_0^{2\pi}\int_0^{\pi}\left[p_0'f'-\tfrac{2}{3}f'^3+\tfrac{1}{2}(\phi_\eta'^2\eta^2+\phi_\theta'^2+\phi_\varphi'^2\csc^2\theta)|_{\eta=1}f'\right.\right. \\
&\left.\left.+\frac{1}{2}\frac{\partial}{\partial\eta}(-\phi_\xi'\eta^2)|_{\eta=1}f'^2\right]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\right\} \\
&+\epsilon^4\left\{\int_0^{2\pi}\int_0^{\pi}\left[p_0'f'^2-\tfrac{1}{8}(f_\theta'^2+f_\varphi'^2\csc^2\theta)^2+\frac{1}{4}\frac{\partial}{\partial\eta}(\phi_\eta'^2\eta^2+\phi_\theta'^2+\phi_\varphi'^2\csc^2\theta)|_{\eta=1}f'^2\right.\right. \\
&\left.\left.+\frac{1}{6}\frac{\partial^2}{\partial\eta^2}(-\phi_\xi'\eta^2)|_{\eta=1}f'^3\right]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\right\} \\
&+O(\epsilon^5).
\end{aligned} \tag{2.4}$$

The linear theory which leads to the frequency relation in (1.1) is recovered from the Lagrangian at $O(\epsilon^2)$. This derivation was outlined in Natarajan & Brown (1986) where it is shown that the effect of a second-order internal resonance could be analysed by calculating the Lagrangian to $O(\epsilon^3)$. The present calculation differs in that at second-order only the bound harmonics of the fundamental frequency are produced and the resonances that appear at third-order require the calculation of the Lagrangian to $O(\epsilon^4)$. The evaluation of the bound harmonics can be carried out as part of the variational procedure and is illustrated in the specific cases that are considered below.

3. Internal resonance of the $n = 3$ normal mode

In order to study the finite-amplitude behaviour of the linear normal mode with $n = 3$, we expand the drop shape and potential as

$$f'(\theta,\varphi,\xi) = S_3(\theta,\varphi,\xi)+\epsilon[S_0(\xi)+S_2(\theta,\varphi,\xi)+S_4(\theta,\varphi,\xi)+S_6(\theta,\varphi,\xi)]+O(\epsilon^2), \tag{3.1a}$$

$$\phi'(\eta,\theta,\varphi,\xi) = \eta^3R_3(\theta,\varphi,\xi)+\epsilon[\eta^2R_2(\theta,\varphi,\xi)+\eta^4R_4(\theta,\varphi,\xi)+\eta^6R_6(\theta,\varphi,\xi)]+O(\epsilon^2), \tag{3.1b}$$

where S_n and R_n are time-dependent surface spherical harmonics of degree n. The solution for ϕ' is chosen to satisfy Laplace's equation in the drop along with the boundary condition that the fluid velocity be finite at the drop centre. The $O(\epsilon)$ terms in (3.1) are the harmonics that are generated through the second-order nonlinearities and their form is anticipated from a successive-approximation procedure. A more general expansion in spherical harmonics could be assumed but it will turn out that the only non-zero terms are the ones that appear above. The term $S_0(\xi)$ in (3.1*a*) is the $O(\epsilon^2)$ correction to the mean radius of the drop; the corresponding term $R_0(\xi)$ has been omitted in (3.1*b*) since only its time derivative appears in the Lagrangian. It is clear from Bernoulli's equation that such a term is equivalent to a time-dependent pressure variation and hence can be absorbed into the $O(\epsilon^2)$ change in the reference pressure defined in (2.3*c*).

We substitute (3.1) into the Lagrangian (2.4) and make use of the orthogonality properties and various other identities involving integrals of products of spherical harmonics and their derivatives given in Appendix A. These manipulations yield

$$
\begin{aligned}
\mathscr{L}' = {} & \epsilon^2 \int_0^{2\pi}\int_0^{\pi} \left\{5S_3^2 + \tfrac{3}{2}R_3^2 - \frac{\partial R_3}{\partial \xi} S_3\right\} \sin\theta \, d\theta \, d\varphi \\
& + \epsilon^4 \Bigg[\left\{4\pi S_0 + \int_0^{2\pi}\int_0^{\pi} S_3^2 \sin\theta \, d\theta \, d\varphi\right\} p_0' - 4\pi S_0^2 + S_0 \int_0^{2\pi}\int_0^{\pi} \left\{-2S_3^2 - 5\frac{\partial R_3}{\partial \xi} + \tfrac{21}{2}R_3^2\right\} \\
& \qquad \times \sin\theta \, d\theta \, d\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \left\{2S_2^2 + R_2^2 - \frac{\partial R_2}{\partial \xi} S_2 + \left(-2S_3^2 - 5\frac{\partial R_3}{\partial \xi} S_3 + 9R_3^2\right) S_2 - 2S_3^2 \frac{\partial R_2}{\partial \xi} + 9R_3 S_3 R_2\right\} \\
& \qquad \times \sin\theta \, d\theta \, d\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \left\{9S_4^2 + 2R_4^2 - \frac{\partial R_4}{\partial \xi} + \left(-2S_3^2 - 5\frac{\partial R_3}{\partial \xi} + \tfrac{11}{2}R_3^2\right) S_4 - 3S_3^2 \frac{\partial R_4}{\partial \xi} + 22R_3 S_3 R_4\right\} \\
& \qquad \times \sin\theta \, d\theta \, d\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \left\{20S_6^2 + 3R_6^2 - \frac{\partial R_6}{\partial \xi} + \left(-2S_3^2 - 5\frac{\partial R_3}{\partial \xi} S_3\right) S_6 - 4S_3^2 \frac{\partial R_6}{\partial \xi} + 39R_3 S_3 R_6\right\} \\
& \qquad \times \sin\theta \, d\theta \, d\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \left\{-\tfrac{1}{8}\mathrm{D}(S_3, S_3)^2 + \tfrac{27}{2}R_3^2 S_3^2 + \tfrac{3}{2}\mathrm{D}(R_3, R_3) S_3^2 - \frac{10}{3}\frac{\partial R_3}{\partial \xi} S_3^2\right\} \sin\theta \, d\theta \, d\varphi \Bigg],
\end{aligned}
\tag{3.2}
$$

where D is a convenient notation for the differential operator acting on pairs of functions such that, for example

$$
\mathrm{D}(S_3, R_3) = \frac{\partial S_3}{\partial \theta}\frac{\partial R_3}{\partial \theta} + \frac{\partial S_3}{\partial \varphi}\frac{\partial R_3}{\partial \varphi} \csc^2\theta. \tag{3.3}
$$

Taking variations of the Lagrangian with respect to p_0' and S_0 yields

$$
4\pi S_0 = -\int_0^{2\pi}\int_0^{\pi} S_3^2 \sin\theta \, d\theta \, d\varphi, \tag{3.4}
$$

$$
4\pi S_0 = -\int_0^{2\pi}\int_0^{\pi} \left(5\frac{\partial R_3}{\partial \xi} S_3 - \tfrac{21}{2}R_3^2\right) \sin\theta \, d\theta \, d\varphi, \tag{3.5}
$$

and these relations may be used to eliminate the corresponding variables from the Lagrangian. The time dependence of the remaining variables is now taken in the form

$$S_3 = S_{3,1}(\xi_2)\,\mathrm{e}^{\mathrm{i}\omega_3\xi} + S^*_{3,1}\,\mathrm{e}^{-\mathrm{i}\omega_3\xi} + O(\epsilon), \tag{3.6a}$$

$$R_3 = -\tfrac{1}{3}\mathrm{i}\omega_3 S_{3,1}(\xi_2)\,\mathrm{e}^{\mathrm{i}\omega_3\xi} + \tfrac{1}{3}\mathrm{i}\omega_3 S^*_{3,1}\,\mathrm{e}^{-\mathrm{i}\omega_3\xi} + O(\epsilon), \tag{3.6b}$$

and for $n = 2, 4, 6$,

$$S_n = S_{n,0} + S_{n,2}\,\mathrm{e}^{2\mathrm{i}\omega_3\xi} + S^*_{n,2}\,\mathrm{e}^{-2\mathrm{i}\omega_3\xi} + O(\epsilon), \tag{3.7a}$$

$$R_n = -2\mathrm{i}\omega_3 R_{n,2}\,\mathrm{e}^{2\mathrm{i}\omega_3\xi} + 2\mathrm{i}\omega_3 R^*_{n,2}\,\mathrm{e}^{-2\mathrm{i}\omega_3\xi} + O(\epsilon), \tag{3.7b}$$

where ω_3 is the dimensionless frequency of the $n = 3$ mode as given by (1.1). The quantities $\{S_{3,1}, S_{n,0}, S_{n,2}, R_{n,2}\}$ are spherical harmonics which depend on the slow timescale ξ_2 (defined below). The first subscript on these quantities denotes the degree of the spherical harmonic while the second term denotes the frequency in multiples of ω_3 of the time-periodic term in the expansion that it multiplies. For brevity, the dependence on the angular variables θ and φ is suppressed in (3.6)–(3.7) as well as in the rest of this section. The primary quantity in the analysis is the complex-valued amplitude $S_{3,1}$ and the remaining quantities are determined in terms of it, as shown below.

The effect of the cubic resonance is to cause the modulation of the amplitude and phase of the primary oscillation mode on a timescale that is $O(\epsilon^2)$ with respect to the primary timescale ξ. This is accounted for in (3.6)–(3.7) by taking $S_{3,1}$ to be a function of the slow timescale ξ_2, defined as

$$\xi_2 = \tfrac{7}{20}\epsilon^2\omega_3\,\xi, \tag{3.8}$$

where the normalization factor in this expression is picked for analytical convenience. Otherwise, the $O(1)$ terms in (3.6) are chosen to be consistent with the linear theory and this may be easily confirmed by substituting into the $O(\epsilon^2)$ terms of the Lagrangian and integrating over a cycle of the oscillation. The vanishing of the integral is consistent with the fact that in a linear system the kinetic energy and the potential energy averaged over an oscillation cycle are equal. The terms in (3.7) are the harmonics generated by the second-order nonlinearities. They have, as expected, frequencies of 0 and $\pm 2\omega_3$ and are composed of all the surface harmonics of even degree up to 6. The mean Lagrangian $\bar{\mathscr{L}}'$, evaluated over a cycle of the primary oscillation is defined as

$$\bar{\mathscr{L}}' = \frac{1}{(2\pi/\omega_3)}\int_0^{2\pi/\omega_3} \mathscr{L}'\,\mathrm{d}\xi, \tag{3.9}$$

and is expressed in terms of the expansions (3.5) and (3.6) by substituting them into (3.2). For the purpose of evaluating this integral, $S_{3,1}$ will vary only slightly over the integration interval and can be taken to be approximately constant. Likewise, time derivatives of $S_{3,1}$ will be $O(\epsilon^2)$ and this ordering must also be preserved in the calculation. A general term in $\mathscr{L}'$ will vary on the primary timescale with a frequency of either 0, $\pm\omega_3$, $\pm 2\omega_3$, $\pm 3\omega_3$ or $\pm 4\omega_3$. The oscillating terms average to zero and isolate the slowly varying terms in the Lagrangian. These slowly varying terms arise solely due to the effect of the internal resonance and hence the averaging procedure effectively emphasizes those terms that govern the modulation behaviour of the oscillatory system.

Much of the tedious algebra is involved in the calculation of $\bar{\mathscr{L}}'$, but even here the terms that yield a non-zero average are easily identified in the lengthy products and the remaining terms are promptly discarded. The final result is

$$
\begin{aligned}
\overline{\mathscr{L}}' = \epsilon^4 \Bigg[& \left\{ \int_0^{2\pi}\int_0^{\pi} \frac{7}{2}\left[\mathrm{i}\frac{\partial S_{3,1}}{\partial \xi_2} S^*_{3,1} - \mathrm{i}\frac{\partial S^*_{3,1}}{\partial \xi^2} S_{3,1} \right] \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right\} \\
& + \frac{1}{4\pi}\left\{ 64\left[\int_0^{2\pi}\int_0^{\pi} |S_{3,1}|^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right]^2 + 172 \int_0^{2\pi}\int_0^{\pi} S^2_{3,1} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right. \\
& \qquad \left. \int_0^{2\pi}\int_0^{\pi} S^{*2}_{3,1} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right\} \\
& + \left\{ \int_0^{2\pi}\int_0^{\pi} [2S^2_{2,0} + 4|S_{2,2}|^2 + 240|R_{2,2}|^2 - 120(R_{2,2}S^*_{2,2} + R^*_{2,2}S_{2,2}) - 44|S_{3,1}|^2 S_{2,0} \right. \\
& \qquad \left. - 82(S^2_{3,1}S^*_{2,2} + S^{*2}_{3,1}S_{2,2}) - 60(S^2_{3,1}R_{2,2} + S^{*2}_{3,1}R^*_{2,2})] \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right\} \\
& + \left\{ \int_0^{2\pi}\int_0^{\pi} [9S^2_{4,0} + 18|S_{4,2}|^2 + 480|R_{4,2}|^2 - 120(R_{4,2}S^*_{4,2} + R^*_{4,2}S_{4,2}) \right. \\
& \qquad \left. - \tfrac{202}{3}|S_{3,1}|^2 S_{4,0} - \tfrac{211}{3}(S^2_{3,1}S^*_{4,2} + S^{*2}_{3,1}S_{4,2}) + 80(S^2_{3,1}R^*_{4,2} + S^{*2}_{3,1}R_{4,2})] \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right\} \\
& + \left\{ \int_0^{2\pi}\int_0^{\pi} [20S^2_{6,0} + 40|S_{6,2}|^2 + 70|R_{6,2}|^2 - 120(\mathrm{R}_{6,2}\mathrm{S}^*_{6,2} + R^*_{6,2}S_{6,2}) - 104|S_{3,1}|^2 \right. \\
& \qquad \left. - 52(S^2_{3,1}S_{6,2} + S^{*2}_{3,1}S_{6,2}) + 300(S^2_{3,1}R^*_{6,2} + S^{*2}_{3,1}R_{6,2})] \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right\} \\
& + \left\{ \int_0^{2\pi}\int_0^{\pi} [-110|S_{3,1}|^4 - \tfrac{1}{4}|\mathrm{D}(S_{3,1}, S_{3,1})|^2 - \tfrac{1}{2}\mathrm{D}(S_{3,1}, S^*_{3,1})^2 - 5\mathrm{D}(S_{3,1}, S_{3,1})S^{*2}_{3,1} \right. \\
& \qquad \left. - 5\mathrm{D}(S^*_{3,1}, S^*_{3,1})S^2_{3,1} + 20\mathrm{D}(S_{3,1}, S^*_{3,1})S_{3,1}S^*_{3,1}] \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \right\} \Bigg].
\end{aligned}
\tag{3.10}
$$

The curly brackets in (3.10) enclose groups of similar significance; there are six such groups that we denote by the Roman numerals I to VI. Here I contains the terms involving the variation of $S_{3,1}$ on the slow timescale ξ_2 while the remaining terms are due to the resonant interaction. Of these, VI consists of terms that arise out of the self interaction of the primary mode while II, III, IV and V consist of terms from the interaction of the primary mode with the bound harmonics of degree 0, 2, 4 and 6, respectively. It is straightforward to determine all the bound harmonics by taking variations of $\overline{\mathscr{L}}'$ with respect to each one of them to yield

$$S_{2,0} = 11|S_{3,1}|^2, \quad S_{2,2} = -2S^2_{3,1}, \quad R_{2,2} = -\tfrac{3}{4}S^2_{3,1}, \tag{3.11}$$

$$S_{4,0} = \tfrac{101}{27}|S_{3,1}|^2, \quad S_{4,2} = -\tfrac{151}{36}S^2_{3,1}, \quad R_{4,2} = -\tfrac{175}{144}S^2_{3,1}, \tag{3.12}$$

$$S_{6,0} = \tfrac{13}{5}|S_{3,1}|^2, \quad S_{6,2} = \tfrac{1}{10}S^2_{3,1}, \quad R_{6,2} = -\tfrac{2}{5}S^2_{3,1}. \tag{3.13}$$

Substituting these results back into the Lagrangian the terms II to V are evaluated as

$$
\begin{aligned}
\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V} = \int_0^{2\pi}\int_0^{\pi} & [64S_{3,1}S^*_{3,1}|_0 S_{3,1}S^*_{3,1} + 172S^2_{3,1}|_0 S^{*2}_{3,1} - 242S_{3,1}S^*_{3,1}|_2 S_{3,1}S^*_{3,1} \\
& + 209S^2_{3,1}|_2 S^{*2}_{3,1} - \tfrac{10201}{81}S_{3,1}S^*_{3,1}|_4 S_{3,1}S^*_{3,1} + \tfrac{21361}{108}S^2_{3,1}|_4 S^{*2}_{3,1} \\
& - \tfrac{676}{5}S_{3,1}S^*_{3,1}|_6 S_{3,1}S^*_{3,1} - \tfrac{626}{5}S^2_{3,1}|_6 S^{*2}_{3,1}] \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi,
\end{aligned}
\tag{3.14}
$$

where the notation $|_i$ indicates that the terms on either side of the bar are coupled only through their parts that are spherical harmonics of degree i. Using the identities (A 5) to (A 7), the self-interaction terms VI are similarly evaluated to yield

$$\text{VI} = \int_0^{2\pi}\int_0^{\pi} [58S_{3,1}S^*_{3,1}|_0 S_{3,1}S^*_{3,1} - 156S^2_{3,1}|_0 S^{*2}_{3,1} + \tfrac{9}{2}S_{3,1}S^*_{3,1}|_2 S_{3,1}S^*_{3,1} - \tfrac{441}{4}S^2_{3,1}|_2 S^{*2}_{3,1}$$
$$-72S_{3,1}S^*_{3,1}|_4 S_{3,1}S^*_{3,1} - 21S^2_{3,1}|_4 S^{*2}_{3,1} - \tfrac{661}{2}S_{3,1}S^*_{3,1}|_6 S_{3,1}S^*_{3,1}$$
$$+\tfrac{279}{4}S^2_{3,1}|_6 S^{*2}_{3,1}]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{3.15}$$

With $\bar{\mathscr{L}}'$ now expressed exclusively in terms of $S_{3,1}$ (and its complex conjugate) the angular integrals are completed by taking

$$S_{3,1}(\theta,\varphi,\xi_2) = \sum_\alpha s_{3,\alpha}(\xi_2)\,C_3^{\alpha *}(\theta,\varphi), \tag{3.16}$$

where $-3 \leqslant \alpha \leqslant 3$. The quantities $\{C_3^\alpha(\theta,\varphi)\}$ are defined in Appendix A and constitute an orthogonal basis for the expansion of an arbitrary spherical harmonic of degree 3. Using (A 8) to (A 12), the averaged Lagrangian $\bar{\mathscr{L}}'$ becomes

$$\bar{\mathscr{L}}' = 4\pi\epsilon^4\left[\sum_\alpha \frac{1}{2}\left\{\mathrm{i}\frac{\mathrm{d}s_{3,\alpha}}{\mathrm{d}\xi_2}s^*_{3,\alpha} - \mathrm{i}\frac{\mathrm{d}s^*_{3,\alpha}}{\mathrm{d}\xi_2}s_{3,\alpha}\right\} + \sum_{\alpha,\beta,\delta,\epsilon}^{\alpha+\beta=\delta+\epsilon} \mathscr{A}^{(3333)}_{\alpha\beta\delta\epsilon}\, s_{3,\alpha}s_{3,\beta}s^*_{3,2\delta}s^*_{3,\epsilon}\right] + O(\epsilon^5), \tag{3.17}$$

where the coefficients appearing above in the interaction term are expressed analytically as the sum of products of $3-j$ symbols (Brink & Satchler 1968). These expressions are given in Appendix B.

Hamilton's principle is applied to (3.17) and the condition for stationarity with respect to a variation in the coefficient $s^*_{3,\lambda}$ is given by

$$\frac{\partial\bar{\mathscr{L}}'}{\partial s^*_{3,\lambda}} - \frac{\mathrm{d}}{\mathrm{d}\xi_2}\left[\frac{\partial\bar{\mathscr{L}}'}{\partial\left(\dfrac{\mathrm{d}s^*_{3,\lambda}}{\mathrm{d}\xi_2}\right)}\right] = 0. \tag{3.18}$$

This expression directly yields the required amplitude equations describing the nonlinear interaction in the form

$$\mathrm{i}\frac{\mathrm{d}s_{3,\lambda}}{\mathrm{d}\xi_2} = \sum_{\alpha,\beta,\gamma}^{\alpha+\beta=\delta+\gamma} \mathscr{B}_{\lambda;\alpha\beta\delta}\, s_{3,\alpha}s_{3,\beta}s^*_{3,\delta}, \tag{3.19}$$

where
$$\mathscr{B}_{\lambda;\alpha\beta\delta} = -(\mathscr{A}^{(3333)}_{\alpha\beta\delta\lambda} + \mathscr{A}^{(3333)}_{\alpha\beta\lambda\delta}). \tag{3.20}$$

By virtue of their definition these coefficients satisfy the symmetry properties

$$\mathscr{B}_{\lambda;\alpha\beta\delta} = \mathscr{B}_{-\lambda;-\alpha-\beta-\delta} = \mathscr{B}_{\lambda;\beta\alpha\delta} = \mathscr{B}_{\delta;\alpha\beta\lambda}, \tag{3.21}$$

which are properties useful in the analysis given below.

A complete analysis of the interaction equations (3.19) is beyond the present scope. Here we consider only the simplest case involving those solutions of (3.19) that have a constant amplitude and a linearly varying phase. These correspond to finite-amplitude time-periodic solutions that have the same form as the small-amplitude solutions of the linear theory, except for a nonlinear correction to the frequency that is proportional to the square of the amplitude. The interaction equations admit only three classes of such solutions; these correspond to (i) the axisymmetric (zonal) harmonic, (ii) an arbitrary linear combination of tesseral harmonics of rank two, and (iii) an arbitrary linear combination of sectorial harmonics. These solutions and their stability properties are investigated below. This nonlinear theory allows only a subset of the class of time-periodic solutions that are admitted by the linear theory. The stability analysis will provide a further restriction by delineating those solutions that are stable to perturbations and hence likely to be observed experimentally.

3.1. *Axisymmetric oscillations*

In the case of axisymmetry the interaction equations reduce to a single equation for the complex amplitude of the zonal harmonic

$$\mathrm{i}\frac{\mathrm{d}s_{3,0}}{\mathrm{d}\xi_2} = \mathscr{B}_{0;000}|s_{3,0}|^2 s_{3,0}, \tag{3.22}$$

which is solved to yield

$$s_{3,0} = |s_{3,0}| \exp\{-\mathrm{i}\mathscr{B}_{0;000}|s_{3,0}|^2 \xi_2\}. \tag{3.23}$$

Using the numerical value of the coefficient in the exponent and making the appropriate substitutions into (3.16) and (3.6) the full solution is given by

$$S_3(\theta,\varphi,\xi) = 2|s_{3,0}| \cos\{(1-3.955\,14|s_{3,0}|^2)\,\omega_3\,\xi\}\,P_3(\cos\theta). \tag{3.24}$$

The form of (3.24) shows the decrease in the oscillation frequency from the linear theory by a value proportional to the square of the amplitude. This is similar to the result obtained by Tsamopoulos & Brown (1984), where a value of $-4.609\,18$ is reported for the proportionality constant compared to the value $-3.955\,14$ given here. The analysis of Tsamopoulos & Brown involves a domain perturbation analysis of the potential-flow equations and boundary conditions. The differences between this approach and the variational formulation used here frustrate even detailed comparisons of intermediate results in order to resolve this discrepancy. The only recourse has been to independently recheck the calculations of the present work for consistency. However, it is clear the difference between the two reported values is not qualitatively significant. The majority of the conclusions regarding the nonlinear dynamical behaviour of drops obtained in this work are based on symmetry considerations involving the spherical harmonics and hence are not crucially affected by the possibility of numerical errors in the evaluation of the coefficients.

The stability of the finite-amplitude axisymmetric oscillations to perturbations in the resonant partners is examined by taking these in the form

$$\begin{bmatrix} s_{3,1}+s_{3,-1} \\ s_{3,2}+s_{3,-2} \\ s_{3,3}+s_{3,-3} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \exp\{-\mathrm{i}\mathscr{B}_{0;000}|s_{3,0}|^2 \xi_2\}, \tag{3.25}$$

where the time-periodic factor (which does not affect the stability arguments) is taken so that the perturbation equations are non-autonomous. The symmetry properties of the coefficients in (3.21) are exploited by summing together pairs of terms corresponding to tesseral harmonics of the same rank and the pair of sectorial harmonics, to decrease the order of the problem. The linearized perturbation equations obtained by this procedure are

$$\mathrm{i}\frac{\mathrm{d}y_n}{\mathrm{d}\xi_2} = \mathscr{B}_{n;00-n}|s_{3,0}|^2 y_n^* + (2\mathscr{B}_{n;n00} - \mathscr{B}_{0;000})|s_{3,0}|^2 y_n \quad \text{for } n = 1,2,3. \tag{3.26}$$

	$\mathscr{B}_{n;00-n}$	$2\mathscr{B}_{n;n00}-\mathscr{B}_{0;000}$
$n=1$	-9.85067	-9.85067
$n=2$	13.90533	-5.63279
$n=3$	-7.01105	-19.42078

TABLE 1. Numerical values of coefficients appearing in equation (3.27)

This system is easily seen to be equivalent to

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}\xi_2^2} = [\mathscr{B}^2_{n;00-n} - (2\mathscr{B}_{n;n00} - \mathscr{B}_{0;000})^2]|s_{3,0}|^4 y_n. \tag{3.27}$$

The signs of the coefficients in the brackets clearly determine the stability of the axisymmetric oscillation to the corresponding perturbations. From the numerical values in table 1, we see that disturbances of the tesseral harmonics of rank one are neutrally stable, those of the sectorial harmonics are stable while the perturbations corresponding to the tesseral harmonics of rank two are unstable.

This resonant instability of the axisymmetric oscillations to non-axisymmetric perturbations with spatial forms having the same degenerate linear frequency provides an explanation for the experimental observations of Trinh & Wang (1982) that were described in §1. Owing to the presence of a suspending fluid, the exact circumstances of their work differs from the theory presented here. Nevertheless, it is clear that the same basic internal resonance mechanism plays a role in their experimental configuration. Indeed, such instabilities may be expected in any nonlinear conservative system executing oscillations about a spherically symmetric base state.

The stability analysis yields the initial shape of the unstable perturbations which grow on a timescale that is proportional to the square of the primary oscillation amplitude. The final dynamical state of the drop due to this instability is not determined by this analysis and requires the study of the full nonlinear set of interaction equations. In real experimental systems the effects of viscosity, however slight, also will become important eventually and damp the drop motions. In the experiments of Trinh & Wang (1982) involving maintained oscillations the effect of viscosity would be to diffuse the mean angular momentum in the surface motions of the growing non-axisymmetric perturbations. This would lead to the non-axisymmetric wavelike motions being transformed into an eventual rigid-body rotation of the drop, as seen in the experiment.

3.2. *Oscillations of the tesseral harmonics of rank two*

The interaction equations that describe the finite-amplitude solutions expressed in terms of the tesseral harmonics of rank two are

$$\mathrm{i}\frac{\mathrm{d}s_{3,2}}{\mathrm{d}\xi_2} = \mathscr{B}_{2;222}|s_{3,2}|^2 s_{3,2} + 2\mathscr{B}_{2;2-2-2}|s_{3,-2}|^2 s_{3,2}, \tag{3.28a}$$

$$\mathrm{i}\frac{\mathrm{d}s_{3,-2}}{\mathrm{d}\xi^2} = \mathscr{B}_{2;222}|s_{3,-2}|^2 s_{3,-2} + 2\mathscr{B}_{2;2-2-2}|s_{3,2}|^2 s_{3,-2}, \tag{3.28b}$$

and these have the solutions

$$s_{3,2} = |s_{3,2}| \exp\{-\mathrm{i}(\mathscr{B}_{2;222}|s_{3,2}|^2 + 2\mathscr{B}_{2;2-2-2}|s_{3,-2}|^2)\,\xi_2\}, \tag{3.29a}$$

$$s_{3,-2} = |s_{3,-2}| \exp\{-\mathrm{i}(\mathscr{B}_{2;222}|s_{3,-2}|^2 + 2\mathscr{B}_{2;2-2-2}|s_{3,2}|^2)\,\xi_2\}. \tag{3.29b}$$

Substituting (3.29) into (3.16) and (3.6) and using numerical values for the coefficients gives for the full solution in the form

$$S_3(\theta, \varphi, \xi) = (\tfrac{1}{30})^{\frac{1}{2}}[|s_{3,2}| \cos\{(1+0.93085|s_{3,2}|^2 - 4.98031|s_{3,-2}|^2)\,\omega_3\,\xi - 2\varphi\} + |s_{3,-2}| \cos\{(1+0.93085|s_{3,-2}|^2 - 4.98031|s_{3,2}|^2)\,\omega_3\,\xi + 2\varphi\}]\,P_3^2(\cos\theta). \tag{3.30}$$

The nature of this solution is made much clearer by going to a frame of reference rotating with a constant angular velocity in the azimuthal direction. The phase angle φ' in this transformed reference frame is

$$\varphi' = \varphi - 1.47779(|s_{3,2}|^2 - |s_{3,-2}|^2)\,\omega_3\,\xi. \tag{3.32}$$

With this substitution (3.30) becomes

$$S_3(\theta, \varphi', \xi) = (\tfrac{1}{30})^{\frac{1}{2}}[|s_{3,2}| \cos\{(1-2.02473\overline{|s_{3,2}|^2 + |s_{3,-2}|^2})\,\omega_3\,\xi - 2\varphi'\} + |s_{3,-2}| \cos\{(1-2.02473\overline{|s_{3,-2}|^2 + |s_{3,2}|^2})\,\omega_3\,\xi + 2\varphi'\}]\,P_3^2(\cos\theta), \tag{3.33}$$

which is seen to describe a superposition of two azimuthally travelling waves of identical form and frequency but with opposite sense. The wave frequency has the familiar nonlinear correction terms that are proportional to the mean value of the sum of the squares of the amplitudes (or equivalently, to the mean energy). This frequency correction is hence non-zero for any non-trivial choice of the initial conditions. The angular velocity of the rotating frame of reference is zero when the mean values of the squares of the amplitudes of the waves are equal in magnitude. This is also equivalent to the condition that the mean angular momentum is zero and no travelling waves are possible in this case, so that (3.33) reduces to

$$S_3(\theta, \varphi', \xi) = (\tfrac{2}{15})^{\frac{1}{2}}|s_{3,2}| \cos\{(1-4.04946|s_{3,2}|^2)\,\omega_3\,\xi\} \cos 2\varphi P_3^2(\cos\theta), \tag{3.34}$$

which is a solution describing a finite amplitude non-axisymmetric standing oscillation in the inertial reference frame.

The stability of the solutions (3.30) to perturbations in resonant partners can be analysed by taking the disturbances to have the form

$$s_{3,0} = y_0 \exp\{-\mathrm{i}(\tfrac{1}{2}\mathscr{B}_{2;222} + \mathscr{B}_{2;2-2-2})(|s_{3,2}|^2 + |s_{3,-2}|^2)\,\xi_2\}, \tag{3.35a}$$

$$(s_{3,1}, s_{3,3}) = (y_1, y_3) \exp\{-\mathrm{i}(\mathscr{B}_{2;222}|s_{3,2}|^2 + 2\mathscr{B}_{2;2-2-2}|s_{3,-2}|^2)\,\xi_2\}, \tag{3.35b}$$

$$(s_{3,-1}, s_{3,-3}) = (y_{-1}, y_{-3}) \exp\{-\mathrm{i}(\mathscr{B}_{2;222}|s_{3,-2}|^2 + 2\mathscr{B}_{2;2-2-2}|s_{3,2}|^2)\,\xi_2\}, \tag{3.35c}$$

where, again, the time-periodic factors appearing above are taken in order to render the perturbation equations non-autonomous.

A suitable parameter for describing the stability results is the quantity α, defined as

$$\alpha = \left\{\frac{|s_{3,-2}|^2 - |s_{3,2}|^2}{|s_{3,-2}|^2 + |s_{3,2}|^2}\right\}^2 \quad (0 \leqslant \alpha \leqslant 1). \tag{3.36}$$

This quantity has the physical interpretation of being the ratio of the square of the mean angular momentum to the square of the mean energy in the primary oscillation (suitably normalized to lie between 0 and 1). There is a unique correspondence between the values of α and all possible solutions of the form (3.30).

The perturbation equation for the zonal harmonic is decoupled from the others and is given by

$$\mathrm{i}\frac{\mathrm{d}y_0}{\mathrm{d}\xi_2} = 2\mathscr{B}_{0;\,-220}|s_{3,\,-2}|\,|s_{3,\,2}|y_0^* + [(2\mathscr{B}_{0;\,022} - \tfrac{1}{2}\mathscr{B}_{2;\,222} + \mathscr{B}_{2;\,2-2-2})\,(|s_{3,\,-2}|^2 + |s_{3,\,2}|^2)\}\,y_0. \tag{3.37}$$

Substituting the numerical values for the coefficients and differentiating gives

$$\frac{\mathrm{d}^2y_0}{\mathrm{d}\xi_2^2} = 193.3446(|s_{3,\,2}|^2 + |s_{3,\,-2}|^2)^2\{1 - 1.00007\alpha\}\,y_0, \tag{3.38}$$

from which it is easily seen that all solutions of the form (3.30) with $\alpha < 0.999929$ are unstable to perturbations in the zonal harmonic.

The equations for the remaining perturbation quantities are

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}y_{\pm 1}}{\mathrm{d}\xi_2} = {}& \{(2\mathscr{B}_{1;\,122} - \mathscr{B}_{2;\,222})|s_{3,\,\pm 2}|^2 + (2\mathscr{B}_{1;\,1-2-2} - 2\mathscr{B}_{2;\,2-2-2})|s_{3,\,\mp 2}|^2\}\,y_{\pm 1} \\ & + 2\mathscr{B}_{1;\,-22-1}|s_{3,\,-2}|\,|s_{3,\,2}|y^*_{\mp 1} + 2\mathscr{B}_{1;\,2-3-2}|s_{3,\,-2}|\,|s_{3,\,2}|y_{\mp 3} + \mathscr{B}_{1;\,223}|s_{3,\,\pm 2}|^2 y^*_{\pm 3},\end{aligned} \tag{3.39a}$$

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}y_{\pm 3}}{\mathrm{d}\xi_2} = {}& \{(\mathscr{B}_{3;\,322} - \mathscr{B}_{2;\,222})|s_{3,\,\pm 2}|^2 + (2\mathscr{B}_{3;\,3-2-2} - 2\mathscr{B}_{2;\,2-2-2})|s_{3,\,\mp 2}|^2\}\,y_{\pm 3} \\ & + 2\mathscr{B}_{3;\,-22-3}|s_{3,\,-2}|\,|s_{3,\,2}|y^*_{\mp 3} + 2\mathscr{B}_{3;\,2-1-2}|s_{3,\,-2}|\,|s_{3,\,2}|y_{\mp 1} + \mathscr{B}_{1;\,223}|s_{3,\,\pm 2}|^2 y^*_{\pm 1}.\end{aligned} \tag{3.39b}$$

The eigenvalues for this set must be obtained from an 8×8 determinant. This calculation was carried out numerically and the results show that solutions of (3.30) with $\alpha \geqslant 0.79$ are unstable to these perturbations.

3.3. *Oscillations of the sectorial harmonics*

The final case studied in this section involves the finite amplitude oscillations of the sectorial harmonics. The interaction equations in this case specialize to

$$\mathrm{i}\frac{\mathrm{d}s_{3,\,3}}{\mathrm{d}\xi_2} = \mathscr{B}_{3;\,333}|s_{3,\,3}|^2 s_{3,\,3} + \mathscr{B}_{3;\,3-3-3}|s_{3,\,-3}|^2 s_{3,\,3}, \tag{3.40a}$$

$$\mathrm{i}\frac{\mathrm{d}s_{3,\,-3}}{\mathrm{d}\xi_2} = \mathscr{B}_{3;\,333}|s_{3,\,-3}|^2 s_{3,\,-3} + 2\mathscr{B}_{3;\,3-3-3}|s_{3,\,3}|^2 s_{3,\,-3}, \tag{3.40b}$$

for which the solutions are given by

$$s_{3,\,3} = |s_{3,\,3}|\,\exp\{-\mathrm{i}(\mathscr{B}_{3;\,333}|s_{3,\,3}|^2 + 2\mathscr{B}_{3;\,3-3-3}|s_{3,\,-3}|^2)\,\xi_2\}, \tag{3.41a}$$

$$s_{3,\,-3} = |s_{3,\,-3}|\,\exp\{-\mathrm{i}(\mathscr{B}_{3;\,333}|s_{3,\,-3}|^2 + 2\mathscr{B}_{3;\,3-3-3}|s_{3,\,3}|^2)\,\xi_2\}. \tag{3.41b}$$

As in the previous case the solution is most conveniently represented in a frame of reference rotating with a constant angular velocity in the azimuthal direction for which the phase angle is given by

$$\varphi' = \varphi - 1.19424(|s_{3,3}|^2 - |s_{3,-3}|^2)\,\omega_3\,\xi, \tag{3.42}$$

and in this frame of reference the solution is given by

$$S_3(\theta,\varphi',\xi) = (\tfrac{1}{180})^{\frac{1}{2}}[-|s_{3,3}|\cos\{(1-5.04099\overline{|s_{3,3}|^2+|s_{3,3}|^2})\,\omega_3\,\xi - 3\varphi'\} + |s_{3,-3}|\cos\{(1-5.04099\overline{|s_{3,3}|^2+|s_{3,-3}|^2})\,\omega_3\,\xi + 3\varphi'\}]\,P_3^3(\cos\theta). \tag{3.43}$$

A special case is the solution with zero mean angular momentum. This corresponds to a standing finite-amplitude oscillation in the inertial reference frame and is given by

$$S_3(\theta,\varphi',\xi) = (\tfrac{1}{45})^{\frac{1}{2}}|s_{3,3}|\sin\{(1-10.08198|s_{3,3}|^2)\,\omega_3\,\xi\}\sin 3\varphi\,P_3^3(\cos\theta). \tag{3.44}$$

The stability of the solutions (3.42) is analysed by taking perturbations of the form

$$\begin{bmatrix} s_{3,0} \\ s_{3,\pm1} \\ s_{3,\pm2} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_{\pm1} \\ y_{\pm2} \end{bmatrix} \exp\{-\mathrm{i}(\tfrac{1}{2}\mathscr{B}_{3,333} + \mathscr{B}_{3;3-3-3})(|s_{3,3}|^2 + |s_{3,-3}|^2)\,\xi_2\}, \tag{3.45}$$

to obtain a non-autonomous set of linearized perturbation equations where the disturbances corresponding to the zonal harmonic and to the tesseral harmonics of various rank all decouple. A similar definition to that in the previous section is adopted for the parameter α,

$$\alpha = \left\{\frac{|s_{3,-3}|^2 - |s_{3,3}|^2}{|s_{3,-3}|^2 + |s_{3,3}|^2}\right\}^2 \quad (0 \leqslant \alpha \leqslant 1). \tag{3.46}$$

Again, there is a unique correspondence between the values of α and all possible solutions of the form (3.43). The details of the stability analysis are very similar to that carried out in §§3.1 and 3.2. The results are that the perturbations corresponding to the zonal harmonic and the tesseral harmonics of rank one are stable and those corresponding to the tesseral harmonics of rank two are unstable for all applicable values of α.

4. Internal resonance of the $n = 2$ and $n = 4$ normal modes

The relation (1.3) between the linear frequencies of the $n = 2$ and $n = 4$ normal modes will lead to these modes being resonantly coupled through the third-order terms in the nonlinearity. The analysis of this resonance is carried out using the same procedure developed in the previous section for the analysis of the $n = 3$ linear normal mode with the only complications arising in this case from the more involved algebra. Thus, the corrections to the drop shape and velocity potential are taken here in the form

$$f'(\theta,\varphi,\xi) = S_2(\theta,\varphi,\xi) + S_4(\theta,\varphi,\xi) + \epsilon[S_0(\xi) + S_6(\theta,\varphi,\xi) + S_8(\theta,\varphi,\xi)] + O(\epsilon^2), \tag{4.1a}$$

$$\phi'(\eta,\theta,\varphi,\xi) = \eta^2 R_2(\theta,\varphi,\xi) + \eta^4 R_4(\theta,\varphi,\xi) + \epsilon[\eta^6 R_6(\theta,\varphi,\xi) + \eta^8 R_8(\theta,\varphi,\xi)] + O(\epsilon^2), \tag{4.1b}$$

where $\{S_n, R_n\}$ are time-dependent surface harmonics of degree n. The form of the $O(\epsilon)$ terms in (4.1) is again anticipated in a successive-approximation procedure and the solution to the velocity potential is consistent with the bounded solution of Laplace's equation. Substituting these expansions into the Lagrangian in (2.4) yields

$$
\begin{aligned}
\mathcal{L}' = \epsilon^2 \Bigg[& \int_0^{2\pi}\int_0^{\pi} \left\{ 2S_2^2 - \frac{\partial R_2}{\partial \xi} S_2 + R_2^2 + 9S_4^2 - 9\frac{\partial R_4}{\partial \xi} S_4 + 2R_4^2 \right\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \Bigg] \\
& + \epsilon^3 \Bigg[\int_0^{2\pi}\int_0^{\pi} \Big\{ -\tfrac{2}{3}(S_2+S_4)^3 - 2\frac{\partial R_2}{\partial \xi}(S_2+S_4)^2 - 3\frac{\partial R_4}{\partial \xi}(S_2+S_4)^2 + \tfrac{7}{2}R_2^2 S_2 \\
& \quad + R_2 R_4(18S_2 + 11S_4) + R_4^2(\tfrac{33}{2}S_2 + 13S_4) \Big\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \Bigg] \\
& + \epsilon^4 \Bigg[\left\{ 4\pi S_0 + \int_0^{2\pi}\int_0^{\pi} (S_2^2 + S_4^2) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \right\} p_0' - 4\pi S_0^2 \\
& + S_0 \int_0^{2\pi}\int_0^{\pi} \left\{ -2S_2^2 - 2S_4^2 - 4\frac{\partial R_2}{\partial \xi} S_2 - 6\frac{\partial R_4}{\partial \xi} S_4 + 5R_2^2 + 18R_4^2 \right\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \Big\{ 20S_6^2 - \frac{\partial R_6}{\partial \xi} S_6 + 3R_6^2 - \Big(4S_2 S_4 + 2S_4^2 + 4\frac{\partial R_2}{\partial \xi} S_4 + 6\frac{\partial R_4}{\partial \xi} S_2 \\
& \quad + 6\frac{\partial R_4}{\partial \xi} S_4 - \tfrac{15}{2}R_4^2 \Big) S_6 \\
& \quad - (8S_2 S_4 + 4S_4^2)\frac{\partial R_6}{\partial \xi} + (26R_2 S_4 + 52R_4 S_2 + 45R_4 S_4) R_6 \Big\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \left\{ 35S_8^2 - \frac{\partial R_8}{\partial \xi} + 4R_8^2 - \left(2S_4^2 + 6\frac{\partial R_4}{\partial \xi} S_4 \right) S_8 - 5S_4^2 \frac{\partial R_8}{\partial \xi} + 68R_4 S_4 R_8 \right\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \left\{ -\tfrac{1}{8}\mathrm{D}(S_2,S_2)^2 + 4R_2^2 S_2^2 + \mathrm{D}(R_2,R_2) S_2^2 - 2\frac{\partial R_2}{\partial \xi} S_2^3 \right\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& \times \int_0^{2\pi}\int_0^{\pi} \left\{ -\tfrac{1}{8}\mathrm{D}(S_4,S_4)^2 + 32R_4^2 S_4^2 + 2\mathrm{D}(R_4,R_4) S_4^2 - 5\frac{\partial R_4}{\partial \xi} S_4^3 \right\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \Big\{ -\tfrac{1}{2}\mathrm{D}(S_2,S_2)\,\mathrm{D}(S_2,S_4) + (8R_2^2 + 2\mathrm{D}(R_2,R_2)) S_2 S_4 \\
& + \left(24R_2 R_4 + 3\mathrm{D}(R_2,R_4) - 6\frac{\partial R_2}{\partial \xi} S_4 - 5\frac{\partial R_4}{\partial \xi} S_2 \right) S_2^2 \Big\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \Big\{ -\tfrac{1}{4}\mathrm{D}(S_2,S_2)\,\mathrm{D}(S_4,S_4) - \tfrac{1}{2}\mathrm{D}(S_2,S_4)^2 + (48R_2 R_4 + 6\mathrm{D}(R_2,R_4)) S_2 S_4 \\
& + \left(4R_2^2 + \mathrm{D}(R_2,R_2) - 6\frac{\partial R_2}{\partial \xi} S_2 \right) S_4^2 + \left(32R_4^2 + 2\mathrm{D}(R_4,R_4) - 15\frac{\partial R_4}{\partial \xi} S_4 \right) S_2^2 \Big\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
& + \int_0^{2\pi}\int_0^{\pi} \Big\{ -\tfrac{1}{2}\mathrm{D}(S_2,S_4)\,\mathrm{D}(S_4,S_4) + \left(64R_4^2 + 4\mathrm{D}(R_4,R_4) - 15\frac{\partial R_4}{\partial \xi} S_4 \right) S_2 S_4 \\
& + \left(24R_2 R_4 + 3\mathrm{D}(R_2,R_4) - 2\frac{\partial R_2}{\partial \xi} S_4 \right) S_4^2 \Big\} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \Bigg]. \qquad (4.2)
\end{aligned}
$$

As in the previous section, similar sets of terms have been grouped together in (4.2) to aid interpretation and to simplify the further analysis. The results of the linear theory are obtained from the $O(\epsilon^2)$ terms while the higher-order terms give the required nonlinear corrections. Taking variations with respect to p_0' and S_0, these variables can then be eliminated from the Lagrangian using the resulting identities.

Since the primary oscillations have frequencies of ω_2 and $\omega_4 = 3\omega_2$, the bound harmonics generated by the second-order correction terms have the frequencies 0, $2\omega_2$, $4\omega_2$ and $6\omega_2$ so that they may be taken in the form

$$S_2 = S_{2,1}(\xi_2)\,e^{i\omega_2\xi} + S^*_{2,1}(\xi_2)\,e^{-i\omega_2\xi} + \epsilon F_2, \tag{4.3a}$$

$$R_2 = -\tfrac{1}{2}i\omega_2 S_{2,1}(\xi_2)\,e^{i\omega_2\xi} + \tfrac{1}{2}i\omega_2 S^*_{2,1}(\xi_2)\,e^{-i\omega_2\xi} + \epsilon G_2, \tag{4.3b}$$

$$S_4 = S_{4,3}(\xi_2)\,e^{i\omega_4\xi} + S^*_{4,3}(\xi_2)\,e^{-i\omega_4\xi} + \epsilon F_4, \tag{4.3c}$$

$$R_4 = -\tfrac{1}{4}i\omega_4 S_{4,3}(\xi_2)\,e^{i\omega_4\xi} + \tfrac{1}{4}i\omega_4 S^*_{4,3}(\xi_2)\,e^{-i\omega_4\xi} + \epsilon G_4, \tag{4.3d}$$

$$S_6 = F_6; \quad R_6 = G_6; \quad S_8 = F_8; \quad R_8 = G_8, \tag{4.3e}$$

where F_n and G_n are given by

$$F_n = S_{n,0} + S_{n,2}\,e^{2i\omega_2\xi} + S^*_{n,2}\,e^{-2i\omega_2\xi} + S_{n,4}\,e^{4i\omega_2\xi} + S^*_{n,4}\,e^{-4i\omega_2\xi} + S_{n,6}\,e^{6i\omega_2\xi} + S^*_{n,6}\,e^{-6i\omega_2\xi}, \tag{4.3f}$$

$$G_n = -2i\omega_2 R_{n,2}\,e^{2i\omega_2\xi} + 2i\omega_2 R^*_{n,2}\,e^{-2i\omega_2\xi} - 4i\omega_2 R_{n,4}\,e^{4i\omega_2\xi} + 4i\omega_2 R^*_{n,4}\,e^{-4i\omega_2\xi} - 6i\omega_2 R_{n,6}\,e^{6i\omega_2\xi} + 6i\omega_2 R^*_{n,6}\,e^{-6i\omega_2\xi}. \tag{4.3g}$$

The leading-order terms in (4.3*a–d*) are chosen to be consistent with the linear theory except for the dependence on the slow timescale. It is convenient in this section to define this slow timescale ξ_2 with a slightly different normalization than the one used in (3.8) as

$$\xi_2 = \tfrac{5}{8}\epsilon^2\omega_2\,\xi. \tag{4.4}$$

The expansions (4.3*a–g*) are substituted into (4.2) and the mean Lagrangian $\overline{\mathscr{L}'}$ is evaluated by averaging over the primary oscillation time period $2\pi/\omega_2$. Then $\overline{\mathscr{L}'}$ is written in terms of the fundamental variables $S_{2,1}$ and $S_{4,3}$ by determining the remaining quantities in (4.3*g*) and (4.3*f*) using the variational procedure. The calculations involved in these two steps are straightforward albeit tedious and we have not ventured to present either the lengthy form of the averaged Lagrangian or the expressions for the several intermediate coefficients. The angular integrals in the averaged Lagrangian are evaluated explicitly by taking expansions for $S_{2,1}$ and $S_{4,3}$ in terms of their corresponding basis functions as

$$S_{2,1}(\theta,\varphi,\xi_2) = \sum_\alpha s_{2,\alpha}(\xi_2)\,C_2^{\alpha^*}(\theta,\varphi), \tag{4.5a}$$

$$S_{4,3}(\theta,\varphi,\xi_2) = \sum_\alpha s_{4,\alpha}(\xi_2)\,C_4^{\alpha^*}(\theta,\varphi). \tag{4.5b}$$

The final result of these manipulations is $\overline{\mathscr{L}'}$ expressed solely in terms of the complex-valued amplitude functions $s_{2,\alpha}$ and $s_{4,\alpha}$ as

$$\begin{aligned}\overline{\mathscr{L}'} = 4\pi\epsilon^4\Bigg[&\sum_\alpha \frac{1}{2}\left\{i\frac{\partial s_{2,\alpha}}{\partial\xi_2}s^*_{2,\alpha} - i\frac{\partial s^*_{2,\alpha}}{\xi_2}s_{2,\alpha}\right\} + \sum_\alpha\frac{5}{12}\left\{i\frac{\partial s_{4,\alpha}}{\partial\xi_2}s^*_{4,\alpha} - i\frac{\partial s^*_{4,\alpha}}{\partial\xi_2}s_{4,\alpha}\right\}\\
&+\sum_{\alpha,\beta,\delta,\epsilon}^{\alpha+\beta=\delta+\epsilon}\mathscr{A}^{(2222)}_{\alpha\beta\delta\epsilon}\,s_{2,\alpha}s_{2,\beta}s^*_{2,\delta}s^*_{2,\epsilon} + \sum_{\alpha,\beta,\delta,\epsilon}^{\alpha+\beta=\delta+\epsilon}\mathscr{A}^{(4444)}_{\alpha\beta\delta\epsilon}\,s_{4,\alpha}s_{4,\beta}s^*_{4,\delta}s^*_{4,\epsilon}\\
&+\sum_{\alpha,\beta,\delta,\epsilon}^{\alpha+\beta=\delta+\epsilon}\mathscr{A}^{(2424)}_{\alpha\beta\delta\epsilon}\,s_{2,\alpha}s_{4,\beta}s^*_{2,\delta}s^*_{4,\epsilon} + \sum_{\alpha,\beta,\delta,\epsilon}^{\alpha+\beta+\delta=\epsilon}\mathscr{A}^{(2224)}_{\alpha\beta\delta\epsilon}\left(s_{2,\alpha}s_{2,\beta}s_{2,\delta}s^*_{4,\epsilon} + s^*_{2,\alpha}s^*_{2,\beta}s^*_{2,\delta}s_{4,\epsilon}\right)\Bigg],\end{aligned} \tag{4.6}$$

where the coefficients appearing in the interaction terms above are again given by sums of products of $3-j$ symbols and whose analytical expressions can be found in Appendix B. The application of Hamilton's principle gives the condition for stationarity with respect to variations in $s_{2,\lambda}$ and $s_{4,\lambda}$ as

$$\frac{\partial \overline{\mathscr{L}}'}{\partial s^*_{2,\lambda}}-\frac{\mathrm{d}}{\mathrm{d}\xi_2}\left[\frac{\partial \overline{\mathscr{L}}'}{\partial\left(\frac{\mathrm{d}s^*_{2,\lambda}}{\mathrm{d}\xi_2}\right)}\right]=\frac{\partial \overline{\mathscr{L}}'}{\partial s^*_{4,\lambda}}-\frac{\mathrm{d}}{\mathrm{d}\xi_2}\left[\frac{\partial \overline{\mathscr{L}}'}{\partial\left(\frac{\mathrm{d}s^*_{4,\lambda}}{\mathrm{d}\xi_2}\right)}\right]=0, \tag{4.7}$$

from which the interaction equations for the internal resonance are found to be

$$\mathrm{i}\frac{\mathrm{d}s_{2,\lambda}}{\mathrm{d}\xi_2}=\sum_{\alpha,\beta,\delta}^{\alpha+\beta=\delta+\lambda}\mathscr{C}_{\lambda;\alpha\beta\delta}\,s_{2,\alpha}s_{2,\beta}s^*_{2,\delta}+\sum_{\alpha,\beta,\delta}^{\alpha+\beta=\delta+\lambda}\mathscr{D}_{\lambda;\alpha\beta\delta}\,s_{2,\alpha}s_{4,\beta}s^*_{4,\beta}+\sum_{\alpha,\beta,\delta}^{\alpha+\beta+\lambda=\delta}\mathscr{E}_{\lambda;\alpha\beta\delta}\,s^*_{2,\alpha}s^*_{2,\beta}s_{4,\delta}, \tag{4.8}$$

$$\mathrm{i}\frac{\mathrm{d}s_{4,\lambda}}{\mathrm{d}\xi_2}=\sum_{\alpha,\beta,\delta}^{\alpha+\beta=\delta+\lambda}\mathscr{F}_{\lambda;\alpha\beta\delta}\,s_{4,\alpha}s_{4,\beta}s^*_{4,\delta}+\sum_{\alpha,\beta,\delta}^{\alpha+\beta=\delta+\lambda}\mathscr{G}_{\lambda;\alpha\beta\delta}\,s_{2,\alpha}s_{4,\beta}s^*_{2,\delta}+\sum_{\alpha,\beta,\lambda}^{\alpha+\beta+\delta=\lambda}\mathscr{H}_{\lambda;\alpha\beta\delta}\,s_{2,\alpha}s_{2,\beta}s_{2,\delta}. \tag{4.9}$$

The values of the coefficients appearing above are

$$\left.\begin{aligned}\mathscr{C}_{\lambda;\alpha\beta\delta}&=-(\mathscr{A}^{(2222)}_{\alpha\beta\lambda\delta}+\mathscr{A}^{(2222)}_{\alpha\beta\delta\lambda}),\\ \mathscr{D}_{\lambda;\alpha\beta\delta}&=-\mathscr{A}^{(2424)}_{\alpha\beta\lambda\delta},\\ \mathscr{E}_{\lambda;\alpha\beta\delta}&=-(\mathscr{A}^{(2224)}_{\alpha\beta\lambda\delta}+\mathscr{A}^{(2224)}_{\alpha\lambda\beta\delta}+\mathscr{A}^{(2224)}_{\lambda\alpha\beta\delta}),\end{aligned}\right\} \tag{4.10}$$

$$\left.\begin{aligned}\mathscr{F}_{\lambda;\alpha\beta\delta}&=-\tfrac{8}{5}(\mathscr{A}^{(4444)}_{\alpha\beta\lambda\delta}+\mathscr{A}^{(4444)}_{\alpha\beta\delta\lambda}),\\ \mathscr{G}_{\lambda;\alpha\beta\delta}&=-\tfrac{8}{5}\mathscr{A}^{(2424)}_{\alpha\beta\delta\lambda},\\ \mathscr{H}_{\lambda;\alpha\beta\delta}&=-\tfrac{8}{5}\mathscr{A}^{(2224)}_{\alpha\beta\delta\lambda}.\end{aligned}\right\} \tag{4.11}$$

The form of the interaction equations (4.8) and (4.9) allows some general conclusions to be drawn regarding the effect of the mode coupling caused by resonance on the long-term dynamics of the two linear normal modes in question. First it is clear that the interaction equations do not admit a constant solution (corresponding to a time-periodic oscillation) that is comprised purely of the components of $n=2$ normal mode. Thus any initial condition involving only this mode will spontaneously degenerate while exciting the components of the $n=4$ normal mode. This takes place through the terms involving the interaction coefficients $\mathscr{H}_{\lambda;\alpha\beta\delta}$. In contrast, the interaction equations permit constant solutions that have only the components of the $n=4$ normal mode. However, the question of the stability and thus the experimental realizability of these solutions requires a more detailed analysis of the interaction equations.

In the following subsections, we have restricted our study to the form and stability of the purely axisymmetric motions. First, we examine the stability of the finite amplitude axisymmetric oscillations of the $n=4$ normal mode. Secondly, we analyse the periodic and quasi-periodic motions that can result due to the combination of the axisymmetric components of the $n=2$ and $n=4$ normal modes.

4.1. *Axisymmetric oscillations of the $n = 4$ normal mode*

The interaction equations for the finite amplitude axisymmetric oscillations of the $n = 4$ normal mode reduce to

$$\mathrm{i}\frac{\mathrm{d}s_{4,0}}{\mathrm{d}\xi_2} = \mathscr{F}_{0;000}|s_{4,0}|^2 s_{4,0}, \tag{4.12}$$

which is integrated to give

$$s_{4,0} = |s_{4,0}| \exp\{-\mathrm{i}\mathscr{F}_{0;000}|s_{4,0}|^2\xi_2\}. \tag{4.13}$$

Using the numerical value of the coefficient $\mathscr{F}_{0;000}$ and substituting into (4.5*a*) and (4.3*a*) yields for the full solution in the form

$$S_4(\theta,\varphi,\xi) = 2|s_{4,0}| \cos\{(1-5.59406|s_{4,0}|^2)\,\omega_4\,\xi\}\,P_4(\cos\theta). \tag{4.14}$$

As in the case of the axisymmetric oscillations of the $n = 3$ linear normal mode studied in the previous section the analysis here predicts a decrease in the frequency of the oscillation proportional to the square of the amplitude. The proportionality constant in this decrement which is -5.59406 above compares with the value of -5.9818 that is reported in Tsamopoulos & Brown (1984).

The stability of the solution in (4.14) to perturbations in the resonant partners may be analysed as follows. The perturbations $s_{2,\lambda}$ of the components of the $n = 2$ mode satisfy the linearized equations

$$\mathrm{i}\frac{\mathrm{d}s_{2,\lambda}}{\mathrm{d}\xi_2} = \mathscr{D}_{\lambda;\lambda 00}|s_{4,0}|^2 s_{2,\lambda}. \tag{4.15}$$

This is easily seen to have only bounded solutions so that the primary oscillations are stable to these perturbations. Reminiscent of the result (3.25) we take the perturbations in the non-axisymmetric components of the $n = 4$ mode to have the form

$$\begin{bmatrix} s_{4,1}+s_{4,-1} \\ s_{4,2}+s_{4,-2} \\ s_{4,3}+s_{4,-3} \\ s_{4,4}+s_{4,-4} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \exp\{-\mathrm{i}\mathscr{F}_{0;000}|s_{4,0}|^2\xi_2\}, \tag{4.16}$$

after exploiting the symmetries of the coefficients $\mathscr{F}_{\lambda;\alpha\beta\delta}$ (which are identical to those of $\mathscr{B}_{\lambda;\alpha\beta\delta}$ in (3.21)) to sum together the perturbation equations corresponding to the tesseral harmonics of the same rank as well as to the sectorial harmonics. The linearized perturbation equations that result are

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}\xi_2^2} = [\mathscr{F}^2_{n;00-n} - (2\mathscr{F}_{n;n00} - \mathscr{F}_{0;000})^2]|s_{4,0}|^4 y_n \quad \text{for } n = 1,2,3,4. \tag{4.17}$$

The signs of the quantities in the brackets determines the stability of the corresponding perturbations. From the numerical values listed in table 2 we see that the disturbances of the tesseral harmonics of rank one are neutrally stable, of the tesseral harmonics of rank three and the sectorial harmonics are stable while the disturbances of the tesseral harmonics of rank two are unstable.

	$\mathscr{F}_{n;\,00-n}$	$2\mathscr{F}_{n;\,n00}-\mathscr{F}_{0;\,000}$
$n=1$	-24.24774	-24.24774
$n=2$	31.72597	-6.07580
$n=3$	-15.64602	-32.135541
$n=4$	14.92950	-44.24937

TABLE 2. Numerical value of the coefficients appearing in equation (4.17)

4.2. *General axisymmetric solutions*

The study of the resonant interaction equations under the restriction to axisymmetric motions can be carried out completely, for it can be shown that in this case (4.13) and (4.14) possess first integrals that reduce the problem to a quadrature (Nayfeh & Mook 1979). It is then possible to examine the nature of the resulting solutions graphically as shown below.

Setting the non-axisymmetric terms in (4.13) and (4.14) to zero, the interaction equations reduce to

$$\mathrm{i}\frac{\mathrm{d}s_{2,0}}{\mathrm{d}\xi_2} = \mathscr{C}_{0;\,000}|s_{2,0}|^2 s_{2,0} + \mathscr{D}_{0;\,000}|s_{4,0}|^2 s_{2,0} + \mathscr{E}_{0;\,000}\, s^{*2}_{2,0}\, s_{4,0}, \tag{4.18a}$$

$$\mathrm{i}\frac{\mathrm{d}s_{4,0}}{\mathrm{d}\xi_2} = \mathscr{F}_{0;\,000}|s_{4,0}|^2 s_{4,0} + \mathscr{G}_{0;\,000}|s_{2,0}|^2 s_{4,0} + \mathscr{H}_{0;\,000}\, s^{3}_{2,0}, \tag{4.18b}$$

where the numerical values of the coefficients are given by (henceforth omitting subscripts)

$$\mathscr{C} = 3.745197,\quad \mathscr{D} = 8.883809,\quad \mathscr{E} = -0.233766,$$

$$\mathscr{F} = 26.85150,\quad \mathscr{G} = 10.660569,\quad \mathscr{H} = -0.0935063.$$

The energy integral is obtained easily from (4.18) as

$$\mathscr{H}|s_{2,0}|^2 + \mathscr{E}|s_{4,0}|^2 = \Gamma, \tag{4.19}$$

where Γ is a constant. The interaction equations are now rewritten in terms of polar variables by defining

$$s_{2,0} = \left(\frac{\Gamma}{\mathscr{H}}\right)^{\frac{1}{2}} x_2\, \mathrm{e}^{\mathrm{i}\theta_2};\quad s_{4,0} = \left(\frac{\Gamma}{\mathscr{E}}\right)^{\frac{1}{2}} x_4\, \mathrm{e}^{\mathrm{i}\theta_4};\quad \xi_2 = \left(\frac{\mathscr{H}}{\mathscr{E}}\right)^{\frac{1}{2}}\frac{\eta_2}{\Gamma};\quad \gamma = 3\theta_2 - \theta_4, \tag{4.20}$$

so that (4.18) is now equivalent to the set

$$-\frac{1}{2}\frac{\mathrm{d}x_2^2}{\mathrm{d}\eta_2} = \frac{1}{2}\frac{\mathrm{d}x_4^2}{\mathrm{d}\eta_2} = x_2^3 x_4 \sin\gamma, \tag{4.21}$$

$$-x_4\frac{\mathrm{d}\gamma}{\mathrm{d}\eta_2} = \alpha_1 x_2^2 x_4 + \alpha_2 x_4^3 + (3x_2 x_4^2 - x_2^3)\cos\gamma, \tag{4.22}$$

where

$$\alpha_1 = (3\mathscr{H} - \mathscr{G})(\mathscr{H}\mathscr{E})^{-\frac{1}{2}} = 3.889317;\ \alpha_2 = (3\mathscr{D} - \mathscr{F})\mathscr{E}^{-2}(\mathscr{H}\mathscr{E})^{\frac{1}{2}} = -0.541298. \tag{4.23}$$

Using the energy integral (4.19) we can rewrite (4.22) in the form

$$\frac{\mathrm{d}}{\mathrm{d}x_4}(x_2^3 x_4 \cos\gamma) = \tfrac{1}{2}\alpha_1 x_4^2 + \tfrac{1}{4}(\alpha_2 - \alpha_1)\, x_4^4, \tag{4.24}$$

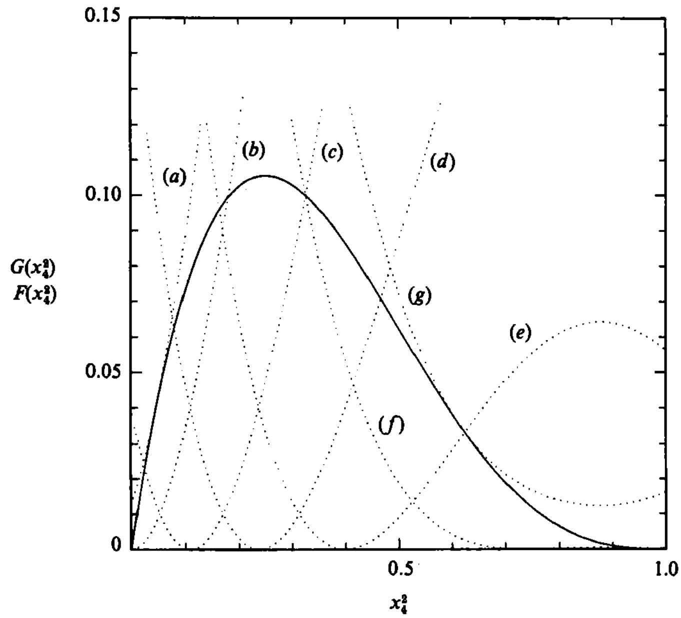

FIGURE 1. The functions F (solid line) and G (dotted lines) as defined in (4.29). The different curves for G correspond to values of C given by (a) 0.112, (b) 0, (c) −0.2, (d) −0.4, (e) −0.6, (f) −0.832, (g) −0.9644.

which is integrated to give

$$x_2^3 x_4 \cos\gamma = \tfrac{1}{2}\alpha_1 x_4^2 + \tfrac{1}{4}(\alpha_2-\alpha_1)\, x_4^4 + C, \tag{4.25}$$

where C is a constant of integration. Substituting for $\sin\gamma$ in (4.21) gives

$$\frac{1}{2}\frac{\mathrm{d}x_4^2}{\mathrm{d}\eta_2} = [(1-x_4^2)^3 x_4^2 - (\tfrac{1}{2}\alpha_1 x_4^2 + \tfrac{1}{4}(\alpha_2-\alpha_1)\, x_4^4 + C)^2]^{\frac{1}{2}}. \tag{4.26}$$

A special class of interesting solutions of (4.26) are those for which there are no amplitude modulations. These solutions are characterized by the fact that there is no energy transfer between the two modes due to the interaction. From (4.21) we see that the necessary condition for this is that the phase of the two oscillating modes must be such that $\sin\gamma = 0$. The amplitudes are not determined by this condition but for consistency must be chosen such that right-hand side of (4.22) is zero. This yields a cubic for the ratio (x_4/x_2) of the form

$$\alpha_2\left(\frac{x_4}{x_2}\right)^3 \pm 3\left(\frac{x_4}{x_2}\right)^2 + \alpha_1\left(\frac{x_4}{x_2}\right) \pm 1 = 0, \tag{4.27}$$

and using the numerical values of α_1 and α_2 the three roots of this cubic are obtained as

$$\left(\frac{x_4}{x_2}\right)^2 = 1.60967, \quad 43.4281, \quad 0.048822. \tag{4.28}$$

The use of the energy integral (i.e. $x_2^2 + x_4^2 = 1$) then yields the amplitudes of the individual modes. The physical interpretation of this result is clear if we consider an initial condition involving the axisymmetric components of both the $n = 2$ and $n = 4$

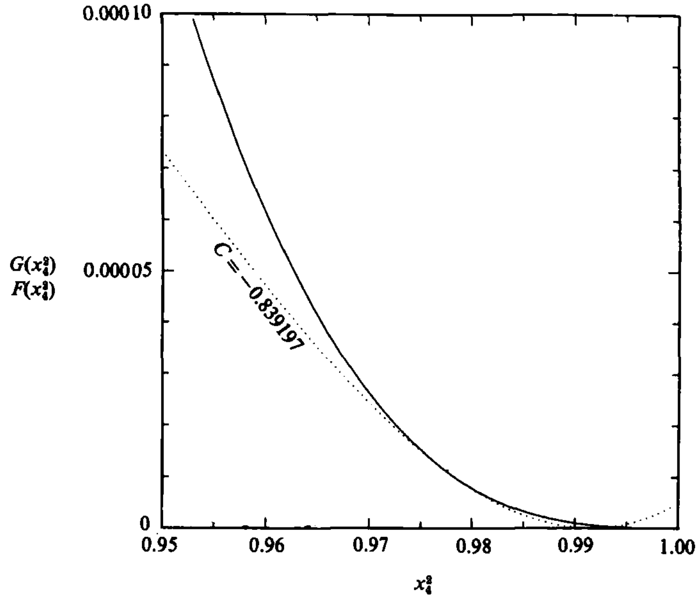

FIGURE 2. The functions F (solid line) and G (dotted line) defined in (4.29). The curve for G corresponds to a value of $C = -0.839197$ and the tangency of the two curves is for the fixed point with $x_4^2 = 0.9775$.

modes with the ratios of their amplitudes being one of the three values in (4.28) and with the individual phase angles being such that $\gamma = 3\theta_2 - \theta_4 = 0$ or π. In this case the resonant interaction will occur without any energy exchange between the two modes. However, the individual modes undergo frequency modulations which are constrained by the requirement that the total phase angle variable γ is invariant.

All other possible initial conditions lead to periodic amplitude modulations. To analyse this general case, we define

$$F(x_4^2) = (1 - xc_4^2)^3 x_4^2; \quad G(x_4^2) = (\tfrac{1}{2}\alpha_1 x_4^2 + \tfrac{1}{4}(\alpha_2 - \alpha_1) x_4^4 + C)^2. \tag{4.29}$$

In order that the radical in (4.26) be positive, we require $F \geqslant G$, so that the values of C are constrained to lie in the interval $-0.9644 \leqslant C \leqslant 0.1127$. The functions $F(x_4^2)$ and $G(x_4^2)$ are plotted for different values of C in figure 1. The intersection points in the curves of these two functions determine the extremal values of the amplitudes during the modulations. For example, we consider the case in which all the initial energy is in the $n = 2$ normal mode. Then $C = 0$ and from figure 1 the extremal values attained by x_4^2 are seen to be 0 and 0.18 (approx.). Thus the initial conditions, which determine the values of the integration constant C and hence the location of the intersection points in figure 1, are seen to play a crucial role in determining the nature of the oscillation. The fixed-point solutions of the interaction equations obtained earlier in (4.28) are obtained at the points where the curves of F and G are tangential to each other for clearly no amplitude modulations are then possible. We can locate the first two fixed-point solutions of (4.28) as lying on the curves labelled a and g in figure 1. The curves corresponding to the third fixed point are tangential at a value of $x_4^2 \approx 0.97749$ and this is shown with better resolution in figure 2.

Phase plane diagrams of (4.26) are also shown in figures 3 and 4 for different values of the integration constant C. These figures provide an indication of the dynamical

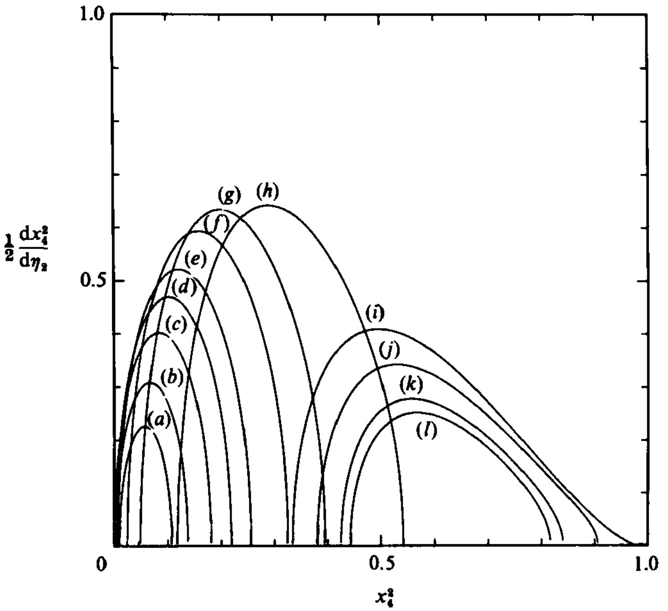

FIGURE 3. A phase plane plot of (4.26) for different values of C corresponding to (*a*) 0.08, (*b*) 0.05, (*c*) 0, (*d*) −0.05, (*e*) −0.1, (*f*) −0.2, (*g*) −0.3, (*h*) −0.5, (*i*) −0.7, (*j*) −0.84, (*k*) −0.88 and (*l*) −0.91.

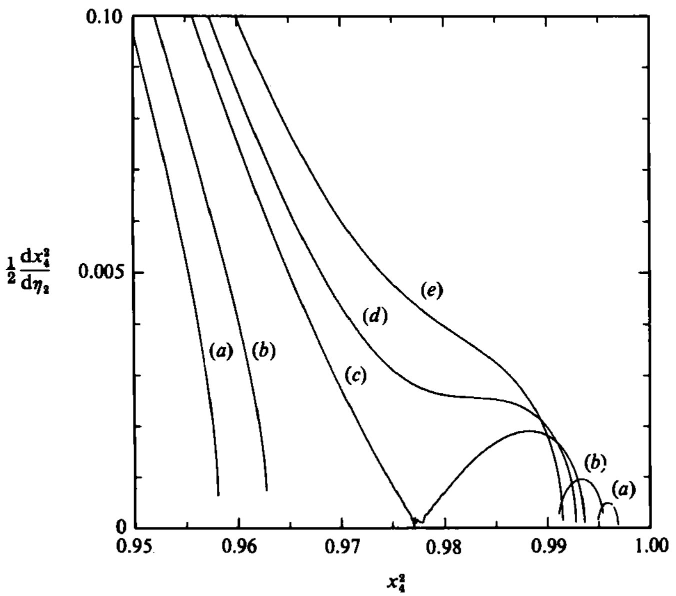

FIGURE 4. Phase plane plot of (4.26) as in figure 3 but with better detail near the fixed point corresponding to $x_4^2 = 0.9775$. The curves correspond to different values of C given by (*a*) −0.838, (*b*) −0.8385, (*c*) −0.8395, (*d*) −0.8398 and (*e*) −0.840.

behaviour that would be expected if the initial conditions were slightly perturbed from the values corresponding to the fixed-point solutions. The two fixed points with the lower values of x_4^2 are stable in the sense that a small change in the value of C results in motions in a small, bounded neighbourhood of the fixed point. This is not true for the remaining fixed-point solution since, as is evident from figure 4, a small change in the value of C leads to considerable amplitude modulations.

The further examination of the stability of these axisymmetric motions to non-axisymmetric perturbations has not been pursued here.

5. Concluding remarks

The amplitude equations that have been derived here for the third-order resonant interactions of the $n = 3$ mode and for the combined interaction of the $n = 2$ and $n = 4$ modes of drop oscillation lead to some interesting properties. In the former case, analysis of the interaction equations shows that simply-periodic finite-amplitude equations are possible that have the usual amplitude-dependent nonlinear frequencies. The spatial forms of these solutions are either axisymmetric, or a superposition of tesseral harmonics of rank two, or a superposition of the sectorial harmonics. In the latter two cases the solutions take the form of rotating waves that are simply periodic only in an appropriate reference frame rotating with a constant angular velocity in the azimuthal direction. However, when the mean angular momentum is zero no rotating waves are possible and these solutions reduce to standing-drop oscillations. The stability analysis showed that none of these solutions were stable.

The absence of stable, simply-periodic finite-amplitude solutions raises the possibility that the eventual long-term drop dynamics will display a three-dimensional, quasi-periodic behaviour with simultaneous amplitude and phase modulation involving several interacting components. To uncover these solutions it is necessary to go beyond the linear-stability calculations to the analysis of the full set of nonlinear amplitude equations. However, in view of the large dimensionality of the resonant interaction set, the prospect of a stochastic long-term behaviour is more likely (Lichtenberg & Lieberman 1983). The main characteristics of this dynamical behaviour would be a sensitive dependence of the solutions on the initial conditions and a uniform Fourier spectrum at low frequencies indicating the absence of any dominant secondary frequencies. The spatial form of the drop associated with this stochastic behaviour would be relatively simple as it would be composed primarily of the components of the $n = 3$ mode along with some negligibly small contributions from the bound harmonics. This stochasticity is even more likely to be exhibited by the amplitude equations describing the combined dynamics of the $n = 2$ and $n = 4$ modes in view of their larger dimensionality.

In the analysis presented here we have assumed an isolated drop and hence take the dynamical effects of the suspending medium to be negligible. Experiments carried out by levitating a drop in the microgravity environment aboard the Space Shuttle make it possible for the requirements of the theory to be closely matched and yet permit the acquisition of data over a sufficient period of time to test our conjectures on the long-term dynamics.

Finally, we remark on the role of viscosity in modifying the effects that have been described in this paper. The inviscid analysis leads to a discontinuity in the tangential velocity at the interface between the drop and the suspending fluid which in real fluids is smoothed out by viscous boundary layers. Lamb (1932) argues that for oscillatory flows the vorticity that is generated at the interface of the drop diffuses inwards but since the flow is periodic it constantly reverses its sign. This constrains the viscous effects to be important only in a thin Stokes boundary layer of thickness $(\nu/\omega)^{\frac{1}{2}}$, where ν and ω are the kinematic viscosity and oscillation frequency, respectively. Since this boundary-layer thickness is small compared to the radius in centimetre-sized water drops, the role of viscosity is confined to damping the free oscillations over a long timescale compared to the oscillation period of the low-order modes. Tsamopoulos & Brown (1984) present a scaling analysis indicating that inviscid resonant effects should be observable above the damping effects of viscosity for the oscillation modes up to $n = 8$.

For liquid drops of greater viscosity the coupling between nonlinearity and viscous effects is a problem that has not been explored. Some fascinating flow-visualization experiments that bear on this problem were conducted by Trinh & Wang (1982) and show a transformation in the flow field from a periodic, nearly potential flow at small oscillation amplitudes, to the appearance at large amplitude of time-dependent eddies that have the spatial symmetry of the primary oscillation. This change in the nature of the flow cannot be explained on the basis of either a linear viscous theory or a nonlinear inviscid theory. The eddies are likely to be due to the generation of a secondary mean flow in the viscous boundary layer. Since the flow is then no longer simply periodic, Lamb's arguments given earlier require modification and the boundary layer can grow. The viscous effects will then manifest throughout the drop leading to the weak secondary recirculating flows.

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Appendix A

This appendix contains some identities involving integrals of products of spherical harmonics. In the following we let S_n, S_m, S_k and S_l denote spherical harmonics of degree n, m, k and l respectively. Then

$$\int_0^{2\pi}\int_0^{\pi} S_n \sin\theta \, d\theta \, d\varphi = 0 \quad \text{for } n \neq 0, \tag{A 1}$$

$$\int_0^{2\pi}\int_0^{\pi} S_n S_m \sin\theta \, d\theta \, d\varphi = 0 \quad \text{for } n \neq m, \tag{A 2}$$

$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial S_n}{\partial\theta}\frac{\partial S_m}{\partial\theta}+\frac{\partial S_n}{\partial\varphi}\frac{\partial S_m}{\partial\varphi}\csc^2\theta\right]\sin\theta \, d\theta \, d\varphi = \begin{cases} 0 & \text{if } n \neq m, \\ n(n+1)\displaystyle\int_0^{2\pi}\int_0^{\pi} S_n S_m \sin\theta \, d\theta \, d\varphi & \text{if } n = m, \end{cases} \tag{A 3}$$

$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial S_n}{\partial\theta}\frac{\partial S_m}{\partial\theta}+\frac{\partial S_n}{\partial\varphi}\frac{\partial S_m}{\partial\varphi}\csc^2\theta\right] S_l \sin\theta \, d\theta \, d\varphi = \tfrac{1}{2}[n(n+1) + m(m+1) - l(l+1)]\int_0^{2\pi}\int_0^{\pi} S_n S_m S_l \sin\theta \, d\theta \, d\varphi. \tag{A 4}$$

The following identities follow directly from (A 1) to (A 4) and are useful in deriving (3.2) and (4.2):

$$\int_0^{2\pi}\int_0^{\pi} S_n S_m S_l S_k \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \sum_{c=0}^{\min(n+m,\,l+k)} \int_0^{2\pi}\int_0^{\pi} S_n S_m|_c S_l S_k \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi, \quad \text{(A 5)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{D}(S_n, S_m) S_l S_k \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \sum_{c=0}^{\min(n+m,\,l+k)} \tfrac{1}{2}[n(n+1) + m(m+1) - c(c+1)] \int_0^{2\pi}\int_0^{\pi} S_n S_m|_c S_l S_k \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi, \quad \text{(A 6)}$$

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{D}(S_n, S_m)\,\mathrm{D}(S_l, S_k) \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \sum_{c=0}^{\min(n+m,\,l+k)} \tfrac{1}{4}[n(n+1) + m)m+1) - c(c+1)] \times [l(l+1) + k(k+1) - c(c+1)] \int_0^{2\pi}\int_0^{\pi} S_n S_m|_c S_l S_k \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \quad \text{(A 7)}$$

The differential operator D appearing in these identities is defined in (3.3) and as defined in the text preceding (3.15) the $|_c$ notation indicates that the terms on either side of the bar are coupled only through their components corresponding to the spherical harmonics of degree c.

The quantities $C_a^\alpha(\theta, \varphi)$ are the basis functions for the spherical harmonics that are defined in Appendix IV of Brink & Satchler (1968) as

$$C_a^\alpha(\theta,\varphi) = (-1)^\alpha \left[\frac{(a-\alpha)!}{(a+\alpha)!}\right]^{\frac{1}{2}} P_a^\alpha(\theta)\,\mathrm{e}^{\mathrm{i}\alpha\varphi} \quad \text{if } \alpha \geqslant 0, \quad \text{(A 8)}$$

$$C_a^{-\alpha}(\theta,\varphi) = (-1)^\alpha C_a^\alpha(\theta,\varphi)^*, \quad \text{(A 9)}$$

where $P_a^\alpha(\theta)$ are the associated Legendre polynomials. The $C_a^\alpha(\theta,\phi)$ satisfy the orthogonality conditions

$$\int_0^{2\pi}\int_0^{\pi} C_a^\alpha(\theta,\varphi)\,C_b^\beta(\theta,\varphi)^* \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{4\pi}{(2a+1)}\delta_{ab}\,\delta_{\alpha\beta}, \quad \text{(A 10)}$$

where δ denotes the Kronecker delta. The product of three basis functions is given in terms of the $3-j$ symbols by Brink & Satchler as

$$\int_0^{2\pi}\int_0^{\pi} C_a^\alpha C_b^\beta C_c^\gamma \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}. \quad \text{(A 11)}$$

The integral of the product of four spherical harmonic basis functions that is used in deriving (3.7) and (4.6) (see Appendix B) then follows from (A 5) as

$$\int_0^{2\pi}\int_0^{\pi} C_a^\alpha C_b^\beta C_d^\delta C_e^\epsilon \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \sum_{c=0}^{\min(a+b,\,c+d)} \int_0^{2\pi}\int_0^{\pi} C_a^\alpha C_b^\beta|_c C_d^\delta C_e^\epsilon \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \sum_{c=0} \mathscr{I}^{(abde)}_{c;\,\alpha\beta\delta\epsilon}. \quad \text{(A 12)}$$

The integrals $\mathscr{I}^{(abde)}_{c;\,\alpha\beta\delta\epsilon}$ can be evaluated in terms of the $3-j$ symbols as

$$\mathscr{I}^{(abde)}_{c;\,\alpha\beta\delta\epsilon} = (-1)^{\alpha+\beta}(2c+1) \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & -\alpha-\beta \end{pmatrix} \begin{pmatrix} d & e & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d & e & c \\ \delta & \epsilon & \alpha+\beta \end{pmatrix}. \quad \text{(A13)}$$

Appendix B

This section contains the analytical expressions for the coefficients appearing in interaction terms of the averaged Lagrangian in (3.7) and (4.6)

$$\mathscr{A}^{(3333)}_{\alpha\beta\delta\epsilon} = (-1)^{\delta+\epsilon}\Big[122\mathscr{I}^{(3333)}_{0;\alpha-\delta\beta-\epsilon} + 16\mathscr{I}^{(3333)}_{0;\alpha\beta-\delta-\epsilon} - \frac{425}{2}\mathscr{I}^{(3333)}_{2;\alpha-\delta\beta-\epsilon} + \frac{395}{4}\mathscr{I}^{(3333)}_{2;\alpha\beta-\delta-\epsilon} - \frac{16033}{81}\mathscr{I}^{(3333)}_{4;\alpha-\delta\beta-\epsilon} + \frac{19093}{108}\mathscr{I}^{(3333)}_{4;\alpha\beta-\delta-\epsilon} - \frac{4657}{10}\mathscr{I}^{(3333)}_{6;\alpha-\delta\beta-\epsilon} - \frac{1109}{20}\mathscr{I}^{(3333)}_{6;\alpha\beta-\delta-\epsilon}\Big], \tag{B1}$$

$$\mathscr{A}^{(2222)}_{\alpha\beta\delta\epsilon} = (-1)^{\delta+\epsilon}\Big[26\mathscr{I}^{(2222)}_{0;\alpha-\delta\beta-\epsilon} + 21\mathscr{I}^{(2222)}_{0;\alpha\beta-\delta-\epsilon} - 73\mathscr{I}^{(2222)}_{2;\alpha-\delta\beta-\epsilon} + \frac{329}{6}\mathscr{I}^{(2222)}_{2;\alpha\beta-\delta-\epsilon} - 108\mathscr{I}^{(2222)}_{4;\alpha-\delta\beta-\epsilon} - \frac{48}{5}\mathscr{I}^{(2222)}_{4;\alpha\beta-\delta-\epsilon}\Big], \tag{B 2}$$

$$\mathscr{A}^{(4444)}_{\alpha\beta\delta\epsilon} = (-1)^{\delta+\epsilon}\Big[380\mathscr{I}^{(4444)}_{0;\alpha-\delta\beta-\epsilon} - 80\mathscr{I}^{(4444)}_{0;\alpha\beta-\delta-\epsilon} - \frac{13553}{32}\mathscr{I}^{(4444)}_{2;\alpha-\delta\beta-\epsilon} + \frac{230129}{2240}\mathscr{I}^{(4444)}_{2;\alpha\beta-\delta-\epsilon} - \frac{8521}{36}\mathscr{I}^{(4444)}_{4;\alpha-\delta\beta-\epsilon} + \frac{45349}{216}\mathscr{I}^{(4444)}_{4;\alpha\beta-\delta-\epsilon} - \frac{37069}{64}\mathscr{I}^{(4444)}_{6;\alpha-\delta\beta-\epsilon} + \frac{55217}{128}\mathscr{I}^{(4444)}_{6;\alpha\beta-\delta-\epsilon} - \frac{9112}{7}\mathscr{I}^{(4444)}_{8;\alpha-\delta\beta-\epsilon} - \frac{3817}{17}\mathscr{I}^{(4444)}_{8;\alpha\beta-\delta-\epsilon}\Big], \tag{B 3}$$

$$\mathscr{A}^{(2424)}_{\alpha\beta\delta\epsilon} = (-1)^{\delta+\epsilon}\Big[220\mathscr{I}^{(2244)}_{0;\alpha-\delta\beta-\epsilon} - \frac{1505}{4}\mathscr{I}^{(2424)}_{2;\alpha-\delta\beta-\epsilon} - 406\mathscr{I}^{(2244)}_{4;\alpha-\delta\beta-\epsilon} + \frac{2182}{5}\mathscr{I}^{(2424)}_{2;\alpha\beta-\delta-\epsilon} + \frac{38861}{56}\mathscr{I}^{(2424)}_{4;\alpha\beta-\delta-\epsilon} - \frac{1887}{7}\mathscr{I}^{(2424)}_{6;\alpha\beta-\delta\epsilon-\epsilon} + 114\mathscr{I}^{(2424)}_{2;\alpha-\epsilon-\delta\beta} - \frac{18029}{40}\mathscr{I}^{(2424)}_{4;\alpha-\epsilon-\delta\beta} - \frac{10992}{13}\mathscr{I}^{(2424)}_{6;\alpha-\epsilon-\delta\beta}\Big], \tag{B 4}$$

$$\mathscr{A}^{(2244)}_{\alpha\beta\delta\epsilon} = (-1)^{\epsilon}\Big[\frac{1772}{2}\mathscr{I}^{(2224)}_{2;\alpha\beta\delta-\epsilon} - \frac{921}{10}\mathscr{I}^{(2224)}_{4;\alpha\beta\delta-\epsilon}\Big]. \tag{B 5}$$

REFERENCES

BRINK, D. M. & SATCHLER, G. R. 1968 *Angular Momentum*, 2nd edn. Clarendon Press.

JACOBI, N., CROONQUIST, A. P., ELLEMAN, D. D. & WANG, T. G. 1982 Acoustically induced oscillation and rotation of a large drop in Space. In *Proc. 2nd Intl Colloq. Drops and Bubbles* (ed. D. H. Le Croissette), p. 31, JPL Publication 82-7, Pasadena.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LICHTENBERG, J. C. & LIEBERMAN, M. A. 1983 *Regular and Stochastic Motion*. Springer.

MARSTON, P. & APFEL, R. E. 1979 Acoustically forced shape oscillation of hydrocarbon drops levitated in water. *J. Colloid Interface Sci.* **68**, 280.

MILLER, C. A. & SCRIVEN, L. E. 1968 The oscillations of a fluid droplet immersed in another fluid. *J. Fluid Mech.* **32**, 417.

NATARAJAN, R. & BROWN, R. A. 1986 Quadratic resonance in the three-dimensional oscillations of drops with surface tension. *Phys. Fluids* **29**, 2788.

NAYFEH, A. H. & MOOK, D. T. 1979 *Nonlinear Oscillations*. John Wiley.

PROSPERETTI, A. 1981 Free oscillations of drops and bubbles: the initial-value problem. *J. Fluid Mech.* **100**, 333.

RAYLEIGH, LORD 1879 On the capillary phenomena of jets. *Proc. R. Soc. Lond.* **29**, 71–97. (*Scientific Papers*, vol. 1, p. 377. Dover.)

REID, W. H. 1960 The oscillations of a viscous liquid drop. *Q. Appl. Maths* **18**, 18.

SIMMONS, W. F. 1969 A variational method for weak resonant wave interactions. *Proc. R. Soc. Lond.* A **309**, 551.

TRINH, E. & WANG, T. G. 1982 Large-amplitude free and driven drop-shape oscillations: experimental observations. *J. Fluid Mech.* **122**, 315.

TRINH, E., ZWERN, A. & WANG, T. G. 1982 An experimental study of small amplitude-drop oscillations in immiscible liquid systems. *J. Fluid Mech.* **115**, 453.

TSAMOPOULOS, J. T. & BROWN, R. A. 1983 Nonlinear oscillations of drops and bubbles. *J. Fluid Mech.* **127**, 519.

TSAMOPOULOS, J. T. & BROWN, R. A. 1984 Resonant oscillations of inviscid drops and bubbles. *J. Fluid Mech.* **147**, 373.

WHITHAM, G. B. 1967 Nonlinear dispersion of water waves. *J. Fluid Mech.* **27**, 399.